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# IMAGE PARAMETERS FOR n-TERMINAL NETWORKS

**Summary:** *The proof of existence and uniqueness of image parameters for n-terminal R-network is given.*

## 1. Introduction.

The title suggests that the subject of this paper is very special. In fact, $n$-terminal network is taken as a well known example of a discrete system with established concept of "image parameters". This concept may be generalized to other systems (continuous, mechanical, etc.) and plays fundamental role in the theory of physical fields. This is why analysis of the concept of "image parameters" is important. This work is a contribution to research on symplectic formulation of physical theories conducted at the Institute of Mathematical Physics in Turin by W.M. Tulczyjew. The aim of this paper is to prove existence and uniqueness of image parameters in the case of reciprocal, passive $n$-terminal $R$-network (i.e. we admit real resistance only).

## 2. The concept of image parameters.

The phase space of an $n$-terminal network is the direct sum $P = \oplus P_k$ of phase spaces associated to each terminal.

$P_k = V_k \oplus I_k$ where $V_k$ is $n_k$ dimensional space of control parameters (voltage), $I_k$ its dual (current). It is possible to describe $P_k$ with so called $s$-parameters:

*Classificazione per soggetto:* AMS (MOS) 1979: 15A24, 78A25, 70C99.

$$v_k^+ = v_k + X_k i_k$$
$$v_k^- = v_k - X_k i_k \quad , \quad v_k \in V_k \quad , \quad i_k \in I_k \ ,$$

$X_k : I_k \to V_k$ is self-adjoint, strictly positively defined. Under this assumptions the mapping

$$P_k \ni (v_k, i_k) \to (v_k^+, v_k^-) \in V_k \oplus V_k \text{ is an isomorphism.} \tag{2.1}$$

Notation: $v^+ = v_1^+ \oplus \ldots \oplus v_n^+ \quad , \quad v^- = v_1^- \oplus \ldots \oplus v_n^-$
$X = X_1 \oplus \ldots \oplus X_n$.

Let a mapping $D$: $V \to I$ give the dynamics of a system. We assume $D$ to be s.a. and positively definite (i.e. the system is reciprocal and passive). The isomorphism (2.1) makes possible describing the dynamics as a graph of a mapping

$$S : V \to V \quad , \quad Sv^+ = v^- .$$

It is easy to check that

$$S = (DX - Id)\,(DX + Id)^{-1}$$

*Definition.* We say that $s$-parameters are "image parameters" iff $(Sv)_k = 0$ when $v = v_1 \oplus \ldots \oplus v_n$ such that $v_i = 0, i \neq k, k = 1, \ldots, n$. (2.2)

In the following section we shall prove

**Theorem 1.**

Let be $D > 0$ , $D = {}^tD$. There exist unique image parameters.

## 3. Proof of the Theorem.

The idea of the proof is due to Prof. S.L. Woronowicz. Let $\mathcal{X}$ be the space of all $s$-parameters, i.e. the space of symmetric, positively definite mappings $X : I \to V$ , $X = X_1 \oplus \ldots \oplus X_n$ , $X_k : I_k \to V_k$. Let us consider the function

$$f : \mathcal{X} \to R \quad , \quad f(X) = Tr\,\log(DXD + 2D + X^{-1}) \tag{3.1}$$

This function has the following properties

i) $f$ is differentiable

ii) at the boundary of $\mathcal{X}$ (i.e. if at least one eigenvalue of $X$ tends to zero or infinity) $f$ tends to infinity.

Thus $f$ has at least one extremal point. In that point $df=0$. On the other hand

$$dTr\log(DXD+2D+X^{-1})=Tr\,d\log(DXD+2D+X^{-1})=$$
$$[Y=DXD+2D+X^{-1}]=Tr\,d\log Y=Tr(Y^{-1}dY).$$

But $Y=(D+X^{-1})X(D+X^{-1})$ , $Y^{-1}=(D+X^{-1})^{-1}X^{-1}(D+X^{-1})^{-1}$ , $dY=DdXD-X^{-1}dXX^{-1}$. Thus $df=Tr(DY^{-1}D-X^{-1}Y^{-1}X^{-1})dX$ and $dY^{-1}D-$ $-X^{-1}Y^{-1}X^{-1}=$

$$=D(D+X^{-1})^{-1}X^{-1}(D+X^{-1})^{-1}D-X^{-1}(D+X^{-1})^{-1}X^{-1}(D+X^{-1})^{-1}X^{-1}=$$
$$=(Id-Q)X^{-1}(Id-Q)^*-QX^{-1}Q^* \quad = \quad /Q=X^{-1}(D+X^{-1})^{-1}/$$
$$=X^{-1}-QX^{-1}-X^{-1}Q^* \quad = \quad X^{-1}-2X^{-1}(D+X^{-1})^{-1}X^{-1}=$$
$$=X^{-1}-X^{-1}(D+X^{-1})^{-1}-[Id-D(D+X^{-1})^{-1}]X^{-1}=$$
$$=(D-X^{-1})(D+X^{-1})^{-1}X^{-1}=(DX-Id)(DX+Id)^{-1}X^{-1}.$$

Thus at an extremal point

$$Tr[(DX-Id)(DX+Id)^{-1}X^{-1}dX]=0 \qquad \forall\, dX-\text{symmetric}.$$

If $dX$ is antisymmetric then $dY$ is antisymmetric. It follows that $Tr\,Y^{-1}dY=0$ and $df=0$ for every $dX$. Hence.

$$(DX-Id)(DX+Id)^{-1}X^{-1} \text{ and } (DX-Id)(DX+Id)^{-1}$$

have property (2.2).

In order to prove uniqueness we need more information on the function $f$.

**Lemma.** The mapping $X\to X^{-1}$ is strictly convex.

*Proof.* We have to prove the inequality

$4(X_1+X_2)^{-1}\leqslant X_1^{-1}+X_2^{-1}$ for $X_i>0$ $i=1,2$ or equivalently
$(X_1+X_2)^{\frac{1}{2}}(X_2^{-1}+X_2^{-1})(X_1+X_2)^{\frac{1}{2}}-4\geqslant 0.$

But $(X_1+X_2)^{\frac{1}{2}}(X_1^{-1}+X_2^{-1})(X_1+X_2)^{\frac{1}{2}}=(X_1+X_2)^{\frac{1}{2}}X_1^{-1}(X_1+X_2)^{\frac{1}{2}}+$ $+(X_1+X_2)^{\frac{1}{2}}X_2^{-1}(X_1+X_2)^{\frac{1}{2}}=A_1+A_2$, where $A_i=(X_1+X_2)^{\frac{1}{2}}X_i^{-1}(X_1+X_2)^{\frac{1}{2}}$.

$A_i\geqslant I$ since $A_1^{-1}+A_2^{-1}=Id$. On the other hand
$A_1^{-1}=Id-A_2^{-1}=A_2^{-1}(A_2-Id)$ , $A_1=(A_2-Id)^{-1}A_2$

Hence $A_1 + A_2 = [A_2(A_2 - Id) + A_2](A_2 - Id)^{-1} = (A_2^2 - Id)(A_2 - Id)^{-1} + (A_2 - Id)^{-1} = Id + A_2 + (A_2 - Id)^{-1}$.

But $A_2 = Id + C$ , $C > 0$ so $Id + A_2 + (A_2 - Id)^{-1} = 2Id + C + C^{-1} = 4 + C^{-1}(C - Id)^2 \geqslant 4$.

If $X_1^{-1} + X_2^{-1} - 4(X_1 + X_2)^{-1} = 0$ then $(X_1 - X_2)X_2^{-1}(X_1 - X_2) = 0$ and $X_1 = X_2$. q.e.d.

Let us consider the function

$$g : \mathscr{X} \to R \quad , \quad g(X) = exp \circ f(X) = det(DXD + 2D + X^{-1}).$$

We know that relations $X_1 \geqslant X_2 > 0$ and $det\, X_1 = det\, X_2$ imply $X_1 = X_2$. It follows that $g$ has exactly one stationary point. Since *exp* is a strictly monotone function also $f$ has exactly one stationary point. q.e.d.

## 4. Consistency theorem.

Suppose $S^{-1}$ exists. According to physical intuition $S^{-1}$ should have the property (2.2).

In fact, we have

**Theorem 2.** Let $DX - Id$ be invertible. $(DX - Id)(DX + Id)^{-1}$ has property (2.2) if and only if $(DX + Id)(DX - Id)^{-1}$ has the same property.

*Proof.* For $X > 0$ we have $DXD - 2D + X^{-1} > 0$.

In fact, $(DX - Id)^2 > 0$ and $(D^{\frac{1}{2}}XD^{\frac{1}{2}} - Id)^2 > 0$. Hence

$$D^{\frac{1}{2}}XD^{\frac{1}{2}} - 2 + D^{-\frac{1}{2}}X^{-1}D^{-\frac{1}{2}} > 0.$$

Now, we can define the function

$$X \to g(X) = Tr\,\log(DXD - 2D + X^{-1}).$$

As in Section 3 we can show that $g$ has exactly one stationary point. At that point, say $X_o$ , $dg = 0$ i.e.
$(DX_o + Id)(DX_o - Id)^{-1}$ satisfies (2.2). Moreover, the function $\det(DXD - 2D + X^{-1})$ has a minimum at $X_o$. It means that

$$DXD - 2D + X^{-1} \geqslant DX_oD - 2D + X_o^{-1} \qquad \text{(not equal).}$$

Hence

$$DXD + 2D + X^{-1} \geqslant DX_o + 2D + X_o^{-1} \qquad \text{(not equal)}$$

and $X_o$ is an extremal point of the function (3.1). q.e.d.

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