Quantum families of maps

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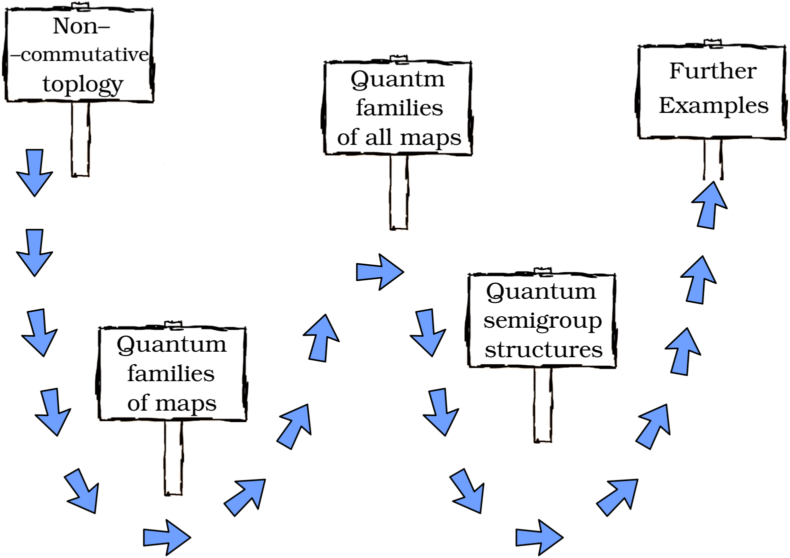
TALK OUTLINE



C*-algebras

Definition

A **C*-algebra** is a Banach algebra A with norm $\|\cdot\|$ and involution

$$\mathsf{A} \ni a \longmapsto a^* \in \mathsf{A},$$

(antilinear & antimultiplicative) such that

$$\|a^*a\| = \|a\|^2$$

for all $a \in \mathsf{A}$.

Examples:

- $\mathrm{B}(\mathscr{H})$ for $\mathscr{H}$ — a Hilbert space,
- $\mathrm{C}(X)$ for X — a compact (Hausdorff) space

or

- $\mathcal{K}(\mathscr{H})$ i.e. the compact operators,
- $\mathrm{C}_0(X)$ for X — locally compact.

Gelfand duality

Theorem

The map

$$\left\{\begin{array}{c}\textit{compact}\\ \textit{spaces}\end{array}\right\} \ni X \longmapsto \mathrm{C}(X) \in \left\{\begin{array}{c}\textit{commutative and}\\ \textit{unital } \mathrm{C}^*\textit{-algebras}\end{array}\right\}$$

extends to an anti-equivalence of categories of

- *compact spaces and continuous maps*

and

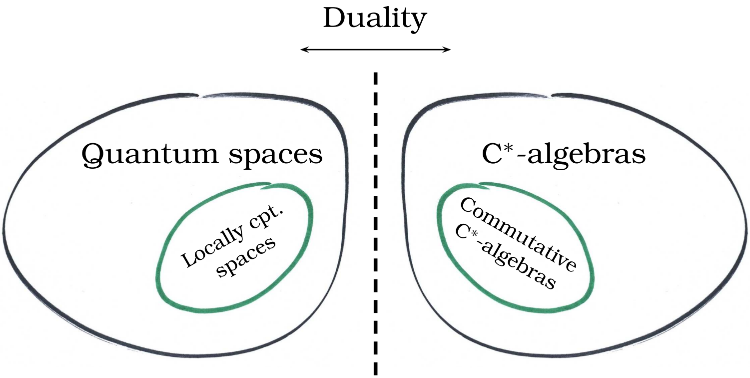
- C^**-algebras and unital* $*$*-homomorphisms.*

- Analogous statement is true for locally compact spaces and all commutative C^*-algebras with appropriate definition of a morphism of C^*-algebras.

Quantum spaces

Definition

A **quantum space** is an object of the category dual to the category of C*-algebras.



Notation & terminology for quantum spaces

- Let $\mathbb{X}$ be a quantum space. The corresponding C^*-algebra will be denoted by $C_0(\mathbb{X})$.
- $\mathbb{X}$ is called **compact** if $C_0(\mathbb{X})$ is unital. In this case we write $C(\mathbb{X})$ for $C_0(\mathbb{X})$.
  From now on we restrict attention to this case.
- $\mathbb{X}$ is called **finite** if $C(\mathbb{X})$ is finite-dimensional (in this case $\mathbb{X}$ is automatically compact).
- Let $\mathbb{X}$ and $\mathbb{Y}$ be quantum spaces. By definition, a continuous map $\mathbb{X} \to \mathbb{Y}$ is a $*$-homomorphism
  $$C(\mathbb{Y}) \longrightarrow C(\mathbb{X}).$$
- If X and Y are compact spaces then
  $$C(X \times Y) \cong C(X) \otimes C(Y)$$
  (minimal tensor product of C^*-algebras).

Classical families of maps

Theorem (J.R. Jackson, 1952)

Let X, Y and E be topological spaces such that X is Hausdorff and E is locally compact. For $\psi \in \mathrm{C}(X \times E, Y)$ define $\sigma(\psi)$ as the mapping from E to $\mathrm{C}(X, Y)$ given by

$$\big[\big(\sigma(\psi)\big)(e)\big](x) = \psi(x, e).$$

Then σ is a homeomorphism of $\mathrm{C}(X \times E, Y)$ onto $\mathrm{C}\big(E, \mathrm{C}(X, Y)\big)$ with all spaces of maps topologized by their respective compact-open topologies.

In other words:

- a (continuous) family of maps $X \to Y$ parametrized by points of E is encoded in a single map $E \to \mathrm{C}(X, Y)$.

What is a quantum family of maps?

Definition

Let $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{E}$ be quantum spaces. A continuous **quantum family of maps** $\mathbb{X} \to \mathbb{Y}$ parametrized by $\mathbb{E}$ is a $*$-homomorphism

$$\Psi\colon\ \mathrm{C}(\mathbb{Y}) \longrightarrow \mathrm{C}(\mathbb{X}) \otimes \mathrm{C}(\mathbb{E}).$$

- If $\mathbb{X} = X$, $\mathbb{Y} = Y$ and $\mathbb{E} = E$ are classical spaces then a quantum family of maps

$$\Psi\colon\ \mathrm{C}(\mathbb{Y}) \longrightarrow \mathrm{C}(\mathbb{X}) \otimes \mathrm{C}(\mathbb{E})$$

  defines uniquely a continuous family of maps $X \to Y$ parametrized by points of E.
- Examples are plentiful!

Composition of quantum families of maps

- Let $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{D}_1$ and $\mathbb{D}_2$ be quantum spaces.
- Consider families of maps

$$\Psi_1\colon \mathrm{C}(\mathbb{X}_2) \longrightarrow \mathrm{C}(\mathbb{X}_1) \otimes \mathrm{C}(\mathbb{D}_1),$$
$$\Psi_2\colon \mathrm{C}(\mathbb{X}_2) \longrightarrow \mathrm{C}(\mathbb{X}_1) \otimes \mathrm{C}(\mathbb{D}_2)$$

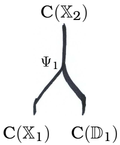
(so $\mathbb{D}_1$ parametrizes maps $\mathbb{X}_1 \to \mathbb{X}_2$ and $\mathbb{D}_2$ parametrizes maps $\mathbb{X}_2 \to \mathbb{X}_3$).
- Define the new quantum family of maps

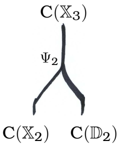
$$\Psi_1 \triangle \Psi_2 = (\Psi_1 \otimes \mathrm{id}) \circ \Psi_2\colon \mathrm{C}(\mathbb{X}_3) \to \mathrm{C}(\mathbb{X}_1) \otimes \mathrm{C}(\mathbb{D}_1) \otimes \mathrm{C}(\mathbb{D}_2).$$

- $\Psi_1 \triangle \Psi_2$ is called the **composition** of Ψ_1 and Ψ_2.
- In classical situation $\Psi_1 \triangle \Psi_2$ corresponds to the family of compositions of all maps from the two families Ψ_1 and Ψ_2.

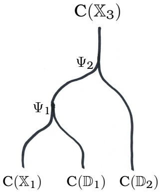
Illustration of composition

- Using graphical notation for Ψ_1 and Ψ_2:





We can represent $\Psi_1 \triangle \Psi_2$ as



- Associativity: $(\Psi_1 \triangle \Psi_2) \triangle \Psi_3 = \Psi_1 \triangle (\Psi_2 \triangle \Psi_3)$.

Quantum families of all maps

Let $\mathbb{X}$ and $\mathbb{Y}$, $\mathbb{E}$ be quantum spaces and let $\Phi\colon \mathrm{C}(\mathbb{Y}) \to \mathrm{C}(\mathbb{X}) \otimes \mathrm{C}(\mathbb{E})$ be a quantum family of maps. We say that

- Φ is the **quantum family of all maps** from $\mathbb{X}$ to $\mathbb{Y}$ and
- $\mathbb{E}$ is the **quantum space of all maps** from $\mathbb{X}$ to $\mathbb{Y}$

if

- for any quantum space $\mathbb{D}$ and
- any quantum family $\Psi\colon \mathrm{C}(\mathbb{Y}) \to \mathrm{C}(\mathbb{X}) \otimes \mathrm{C}(\mathbb{D})$

there exists a unique $\Lambda\colon \mathrm{C}(\mathbb{E}) \to \mathrm{C}(\mathbb{D})$ such that

$$\begin{array}{ccc} \mathrm{C}(\mathbb{Y}) & \xrightarrow{\quad\Phi\quad} & \mathrm{C}(\mathbb{X}) \otimes \mathrm{C}(\mathbb{E}) \\ \Big\| & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\Lambda} \\ \mathrm{C}(\mathbb{Y}) & \xrightarrow{\quad\Psi\quad} & \mathrm{C}(\mathbb{X}) \otimes \mathrm{C}(\mathbb{D}) \end{array}$$

EXISTENCE

- The quantum space of all maps $\mathbb{X} \to \mathbb{Y}$ often does not exists
(or rather, it is not locally compact).
- In 1979 S.L. Woronowicz stated

THEOREM

Let $\mathbb{X}$ and $\mathbb{Y}$ be quantum spaces such that $\mathrm{C}(\mathbb{X})$ is finite dimensional and $\mathrm{C}(\mathbb{Y})$ is finitely generated and unital. Then the quantum space of all maps $\mathbb{X} \to \mathbb{Y}$ exists. Moreover this quantum space is compact.

- When the quantum space of all maps from $\mathbb{X}$ to $\mathbb{Y}$ exists, it is unique.
- Very interesting case: $\mathbb{X} = \mathbb{Y} = \mathbb{M}$ with $\mathrm{C}(\mathbb{M})$ finite-dimensional.

Example 1

- Let $\mathbb{M}$ be the classical two point space:

$$\mathbb{M} = \{\bullet, \bullet\}$$

(i.e. $C(\mathbb{M}) = \mathbb{C}^2$).

- The **classical** space o all maps $\mathbb{M} \to \mathbb{M}$ is $\{\bullet, \bullet, \bullet, \bullet\}$.
- The **quantum** space $\mathbb{E}$ of all maps $\mathbb{M} \to \mathbb{M}$ is such that

$$C(\mathbb{E}) = \left\{ f \in C\big([0,1], M_2(\mathbb{C})\big) \,\middle|\, f(0), f(1) \text{ are diagonal} \right\}.$$

- The quantum family of all maps $\mathbb{M} \to \mathbb{M}$ is $\Phi\colon \mathbb{C}^2 \to \mathbb{C}^2 \otimes C(\mathbb{E})$

$$\Phi\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \otimes P + \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \otimes Q,$$

where

$$P(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q(t) = \tfrac{1}{2} \begin{bmatrix} 1 - \cos 2\pi t & \mathrm{i} \sin 2\pi t \\ -\mathrm{i} \sin 2\pi t & 1 + \cos 2\pi t \end{bmatrix}.$$

Quantum semigroup structure

- Let $\mathbb{M}$ be a finite quantum space.
- Let $\mathbb{E}$ be the quantum space of all maps $\mathbb{M} \to \mathbb{M}$ and let
$$\Phi\colon\ \mathrm{C}(\mathbb{M}) \longrightarrow \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{E})$$
be the quantum family of all maps $\mathbb{M} \to \mathbb{M}$.
- The universal property of $(\mathbb{E}, \Phi)$ gives a
$$\Delta\colon\ \mathrm{C}(\mathbb{E}) \longrightarrow \mathrm{C}(\mathbb{E}) \otimes \mathrm{C}(\mathbb{E})$$
such that $\Phi \vartriangle \Phi = (\mathrm{id} \otimes \Delta) \circ \Phi$:

$$\begin{array}{ccc}
\mathrm{C}(\mathbb{M}) & \xrightarrow{\ \Phi\ } & \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{E}) \\
\Big\| & & \Big\downarrow \mathrm{id}\otimes\Delta \\
\mathrm{C}(\mathbb{M}) & \xrightarrow{\ \Phi\vartriangle\Phi\ } & \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{E}) \otimes \mathrm{C}(\mathbb{E})
\end{array}$$

PROPERTIES

THEOREM

Let $\mathbb{M}$ *be a finite quantum space and let* $\mathbb{E}$ *be the quantum space of all maps* $\mathbb{M} \to \mathbb{M}$. *Let*

$$\Phi\colon \mathrm{C}(\mathbb{M}) \longrightarrow \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{E}),$$
$$\Delta\colon \mathrm{C}(\mathbb{E}) \longrightarrow \mathrm{C}(\mathbb{E}) \otimes \mathrm{C}(\mathbb{E})$$

be as above. Then

1. Δ *is* ***coassociative***: $(\Delta \otimes \mathrm{id})\circ\Delta = (\mathrm{id} \otimes \Delta)\circ\Delta$.
2. *There exists a unique character* ϵ *of* $\mathrm{C}(\mathbb{E})$ *such that*

$$(\mathrm{id} \otimes \epsilon)\circ\Phi = \mathrm{id}.$$

3. *We have:* $(\mathrm{id} \otimes \epsilon)\circ\Delta = (\epsilon \otimes \mathrm{id})\circ\Delta = \mathrm{id}$.
4. *The spectrum of* $\mathrm{C}(\mathbb{M})$ *coincides with the compact space of* $*$*-homomorphisms* $\mathrm{C}(\mathbb{M}) \to \mathrm{C}(\mathbb{M})$.

Example 1 continued

- For $C(\mathbb{M}) = \mathbb{C}^2$ the quantum space $\mathbb{E}$ of all maps $\mathbb{M} \to \mathbb{M}$ is

$$C(\mathbb{E}) = \left\{ f \in C\big([0,1], M_2(\mathbb{C})\big) \,\middle|\, f(0), f(1) \text{ are diagonal} \right\}.$$

- We have

$$\Delta(P) = P \otimes P + (\mathbb{1} - P) \otimes Q, \quad \Delta(Q) = Q \otimes P + (\mathbb{1} - Q) \otimes Q.$$

(Recall: $P(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q(t) = \frac{1}{2}\begin{bmatrix} 1-\cos 2\pi t & \mathrm{i}\sin 2\pi t \\ -\mathrm{i}\sin 2\pi t & 1+\cos 2\pi t \end{bmatrix}$.)

- The **co-unit** ϵ is given by

$$\epsilon(P) = 1, \qquad \epsilon(Q) = 0.$$

- $\mathbb{E}$ is <u>not</u> a quantum <u>group</u>.

Quantum groups

Definition

- A compact quantum space $\mathbb{G}$ is a **compact quantum semigroup** if there exists $\Delta\colon \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$ such that $(\Delta \otimes \mathrm{id})\circ\Delta = (\mathrm{id} \otimes \Delta)\circ\Delta$.
- $\mathbb{G}$ is called a **compact quantum group** if

$$\overline{\Delta\bigl(\mathrm{C}(\mathbb{G})\bigr)\bigl(\mathbb{1} \otimes \mathrm{C}(\mathbb{G})\bigr)} = \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G}),$$
$$\overline{\bigl(\mathrm{C}(\mathbb{G}) \otimes \mathbb{1}\bigr)\Delta\bigl(\mathrm{C}(\mathbb{G})\bigr)} = \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G}).$$

- If $\mathbb{M}$ is a finite quantum space and $\mathbb{E}$ is the quantum semigroup of all maps $\mathbb{M} \to \mathbb{M}$ then $\mathbb{E}$ is not a quantum group unless $\mathbb{M} = \{\bullet\}$.
- If $\mathbb{M} = \{\bullet, \bullet\}$ then $\mathbb{E}$ <u>is</u> a quantum group with:

$$\Delta(P) = (P - \mathbb{1}) \otimes P + \mathbb{1} \otimes \mathbb{1} + P \otimes (P - \mathbb{1}),$$
$$\Delta(Q) = (Q - \mathbb{1}) \otimes Q + \mathbb{1} \otimes \mathbb{1} + Q \otimes (Q - \mathbb{1}).$$

DIGRESSION: QUANTUM GROUPS

- Compact quantum groups were defined by S.L. Woronowicz in 1987.
- They have Haar measures.
- Appropriate notion of representations can be introduced.
- The Peter-Weyl theory has been developed in full generality.
- Examples include:
  - quantum deformations like $S_qU(2)$,
  - "free" compact quantum groups,
  - quantum isometry groups of spectral triples,
  - many more...
- **Locally** compact quantum groups are object of current research.
- Theory of actions of quantum groups on quantum spaces is being studied (many mysteries still to be solved there).

Semigroup of all maps — summary

- Let $\mathbb{M}$ be a finite quantum space.
- Let $\mathbb{E}$ be the quantum space of all maps $\mathbb{M} \to \mathbb{M}$.
- $\mathbb{E}$ carries a canonical stricture of a compact quantum semigroup.
- The quantum family of all maps $\mathbb{M} \to \mathbb{M}$

  $$\Phi\colon \mathrm{C}(\mathbb{M}) \longrightarrow \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{E})$$

  is an action of $\mathbb{E}$ on $\mathbb{M}$:

  $$(\Phi \otimes \mathrm{id}) \circ \Phi = (\mathrm{id} \otimes \Delta) \circ \Phi.$$

- Classical analogy:
  - M — space,
  - E — semigroup of all maps $M \to M$.
  - $\phi\colon M \times E \to M$ describes the action: $\phi(m, \lambda) = \lambda(m)$.

  Then Φ corresponds to the map

  $$\mathrm{C}(M) \ni f \longmapsto f \circ \phi \in \mathrm{C}(M \times E).$$

Quantum families preserving a state

- Let M be a finite space and let μ be a measure on M.
- Let E be the semigroup of all maps $M \to M$.
- The set of all maps $M \to M$ preserving μ is a subsemigroup of E.

Definition

Let $\mathbb{M}$ be a finite quantum space and let ω be a state on $\mathrm{C}(\mathbb{M})$ (positive linear functional of norm 1). Let $\mathbb{D}$ be another quantum space and let $\Psi\colon\ \mathrm{C}(\mathbb{M}) \to \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{D})$ be a quantum family of maps $\mathbb{M} \to \mathbb{M}$.
We say that Ψ **preserves** ω if

$$(\omega \otimes \mathrm{id})\big(\Psi(x)\big) = \omega(x)\mathbb{1},$$

for all $x \in \mathrm{C}(\mathbb{M})$.

Quantum semigroup preserving a state

Theorem

Let $\mathbb{M}$ *be a finite q. space and* ω *a state on* $\mathrm{C}(\mathbb{M})$*. Then*

- *there exists a unique quantum family*

$$\Phi_\omega\colon\ \mathrm{C}(\mathbb{M}) \longrightarrow \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{W})$$

such that for any quantum family

$$\Psi\colon\ \mathrm{C}(\mathbb{M}) \longrightarrow \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{D})$$

preserving ω *there exists a unique* $\Lambda\colon\ \mathrm{C}(\mathbb{W}) \to \mathrm{C}(\mathbb{D})$ *such that*

$$\begin{array}{ccc} \mathrm{C}(\mathbb{M}) & \xrightarrow{\ \Phi_\omega\ } & \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{W}) \\ \Big\| & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\Lambda} \\ \mathrm{C}(\mathbb{M}) & \xrightarrow{\ \Psi\ } & \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{D}) \end{array}$$

- Φ_ω *preserves* ω*,*
- $\mathbb{W}$ *is a compact quantum semigroup (canonically).*

EXAMPLE

- Let $\mathrm{C}(\mathbb{M}) = M_2(\mathbb{C})$, $\omega_q\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right) = \frac{a+q^2 d}{1+q^2}$ $\quad (q \in]0,1])$.
- The quantum semigroup $\mathbb{W}$ of all maps $\mathbb{M} \to \mathbb{M}$ preserving ω_q looks as follows:
  - $\mathrm{C}(\mathbb{W})$ is generated by β, γ and δ s.t.

$$\begin{aligned} q^4\delta^*\delta + \gamma^*\gamma + q^4\delta\delta^* + \beta\beta^* &= \mathbb{1}, & \beta\gamma &= -q^4\delta^2, \\ \beta^*\beta + \delta^*\delta + \gamma\gamma^* + \delta\delta^* &= \mathbb{1}, & \gamma\beta &= -\delta^2, \\ \gamma^*\delta - q^2\delta^*\beta + \beta\delta^* - q^2\delta\gamma^* &= 0, & \beta\delta &= q^2\delta\beta, \\ q^4\delta\delta^* + \beta\beta^* + q^2\gamma\gamma^* + q^2\delta\delta^* &= \mathbb{1}, & \delta\gamma &= q^2\gamma\delta \\ q^4\delta^*\delta + \gamma^*\gamma + q^2\beta^*\beta + q^2\delta^*\delta &= q^2\mathbb{1}. \end{aligned}$$

  - The comultiplication $\Delta\colon \mathrm{C}(\mathbb{W}) \to \mathrm{C}(\mathbb{W}) \otimes \mathrm{C}(\mathbb{W})$ is

$$\begin{aligned} \Delta(\beta) &= q^4\delta\gamma^* \otimes \delta - q^2\beta\delta^* \otimes \delta + \beta \otimes \beta + \gamma^* \otimes \gamma - q^2\delta^*\beta \otimes \delta + \gamma^*\delta \otimes \delta, \\ \Delta(\gamma) &= q^4\gamma\delta^* \otimes \delta - q^2\delta\beta^* \otimes \delta + \gamma \otimes \beta + \beta^* \otimes \gamma - q^2\beta^*\delta \otimes \delta + \delta^*\gamma \otimes \delta, \\ \Delta(\delta) &= -q^2\gamma^*\gamma \otimes \delta - q^2\delta\delta^* \otimes \delta + \delta \otimes \beta + \delta^* \otimes \gamma + \beta^*\beta \otimes \delta + \delta^*\delta \otimes \delta. \end{aligned}$$

  - The counit ϵ maps γ and δ to 0 and β to 1.

A small improvement

- Consider $\mathbb{M}$ with $\mathrm{C}(\mathbb{M}) = M_2(\mathbb{C})$ and ω_q as before.
- The semigroup $\mathbb{W}$ described above contains the largest quantum group preserving ω_q.
- This quantum group turns out to be the **quantum SO(3) group** defined in 1989 by P. Podleś via representation theory.
- $\mathrm{C}\big(\mathrm{S}_q\mathrm{O}(3)\big)$ was originally known to be generated by A, C, G, K, L satisfying

$$
\begin{aligned}
L^*L &= (\mathbb{1}-K)(\mathbb{1}-q^{-2}K), & CC^* &= q^2K - q^4K^2, & AG &= q^2GA,\\
LL^* &= (\mathbb{1}-q^2K)(\mathbb{1}-q^4K), & LK &= q^4KL, & AC &= CA,\\
G^*G &= GG^*, & GK &= KG, & LG^* &= q^4G^*L,\\
K^2 &= G^*G, & AK &= q^2KA, & A^2 &= q^{-1}LG,\\
A^*A &= K - K^2, & CK &= q^2KC, & A^*L &= q^{-1}(\mathbb{1}-K)C,\\
AA^* &= q^2K - q^4K^2, & LG &= q^4GL, & K^* &= K.\\
C^*C &= K - K^2, & LA &= q^2AL, & &
\end{aligned}
$$

QUANTUM COMMUTANTS

- Let M be a finite space and let $\mathscr{F}$ be a family of maps $M \to M$.
- Let E be the semigroup of all maps $M \to M$.
- The set of all maps $M \to M$ commuting with elements of $\mathscr{F}$ is a subsemigroup of E.

DEFINITION

Let $\mathbb{M}$ be a finite quantum space and let

$$\Psi_1\colon \mathrm{C}(\mathbb{M}) \to \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{D}_1), \qquad \Psi_2\colon \mathrm{C}(\mathbb{M}) \to \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{D}_2)$$

be two quantum families of maps. We say that Ψ_1 and Ψ_2 **commute** if

$$(\mathrm{id} \otimes \sigma) \circ (\Psi_1 \triangle \Psi_2) = \Psi_2 \triangle \Psi_1,$$

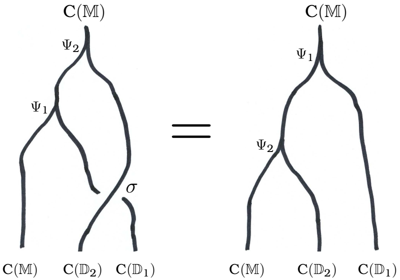
where σ is the flip $\mathrm{C}(\mathbb{D}_1) \otimes \mathrm{C}(\mathbb{D}_2) \to \mathrm{C}(\mathbb{D}_2) \otimes \mathrm{C}(\mathbb{D}_1)$.

Illustration of commutation

Quantum families

$$\Psi_1 : \ C(\mathbb{M}) \longrightarrow C(\mathbb{M}) \otimes C(\mathbb{D}_1),$$
$$\Psi_2 : \ C(\mathbb{M}) \longrightarrow C(\mathbb{M}) \otimes C(\mathbb{D}_2)$$

commute if



Quantum commutant

Theorem

Let $\mathbb{M}$ *be a finite q. space and* $\Psi\colon \mathrm{C}(\mathbb{M}) \to \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{D})$ *a quantum family of maps* $\mathbb{M} \to \mathbb{M}$*. Then*

- *there exists a unique quantum family*

  $$\Phi_\Psi\colon \mathrm{C}(\mathbb{M}) \longrightarrow \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{U})$$

  such that for any quantum family $\Theta\colon \mathrm{C}(\mathbb{M}) \to \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{P})$ *commuting with* Ψ *there exists a unique* $\Lambda\colon \mathrm{C}(\mathbb{U}) \to \mathrm{C}(\mathbb{P})$ *such that*

  $$\begin{array}{ccc} \mathrm{C}(\mathbb{M}) & \xrightarrow{\ \Phi_\Psi\ } & \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{U}) \\ \| & & \downarrow{\scriptstyle \mathrm{id}\otimes\Lambda} \\ \mathrm{C}(\mathbb{M}) & \xrightarrow{\ \Theta\ } & \mathrm{C}(\mathbb{M}) \otimes \mathrm{C}(\mathbb{P}) \end{array}$$

- Φ_Ψ *commutes with* Ψ*,*
- $\mathbb{U}$ *is a compact quantum semigroup (canonically).*

Example

- As before let $\mathbb{M}$ be such that $\mathrm{C}(\mathbb{M}) = M_2(\mathbb{C})$.
- Let $\mathbb{U}$ be the commutant of the (classical) family of maps $\mathbb{M} \to \mathbb{M}$ consisting of the single automorphism of $\mathrm{C}(\mathbb{M})$:

$$\psi\colon \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} d & c \\ b & a \end{bmatrix}.$$

- This family is described in our language by

$$\Psi\colon \mathrm{C}(\mathbb{M}) \longrightarrow \mathrm{C}(\mathbb{M}) \otimes \mathbb{C}$$

  given by $\Psi(m) = \psi(m) \otimes 1$.
- Let $\mathbb{U}$ be the quantum commutant of Ψ.

EXAMPLE

- The C*-algebra $C(\mathbb{U})$ is generated by α, β and γ with

$$\beta = \beta^*, \quad \gamma = \gamma^*$$

and

$$\begin{aligned} \alpha^*\alpha + \gamma^2 + \alpha\alpha^* + \beta^2 &= \mathbb{1}, & \alpha^2 + \beta\gamma &= 0, \\ \alpha^*\beta + \gamma\alpha^* + \alpha\gamma + \beta\alpha &= 0, & \alpha\beta + \beta\alpha^* &= 0, \\ \gamma\alpha + \alpha^*\gamma &= 0. \end{aligned}$$

- The comultiplication is

$$\begin{aligned} \Delta(\alpha) &= \mathbb{1} \otimes \alpha + (\alpha^*\alpha + \gamma^2) \otimes (\alpha^* - \alpha) + \alpha \otimes \beta + \alpha^* \otimes \gamma, \\ \Delta(\beta) &= (\alpha\gamma + \beta\alpha) \otimes (\alpha - \alpha^*) + \beta \otimes \beta + \gamma \otimes \gamma, \\ \Delta(\gamma) &= (\beta\alpha + \alpha\gamma) \otimes (\alpha^* - \alpha) + \gamma \otimes \beta + \beta \otimes \gamma, \end{aligned}$$

- $\mathbb{U}$ is not a compact quantum group (with this Δ).

All maps into a quantum semigroup

- Let S be a finite set and $\mathbb{S}$ a quantum semigroup.
- The quantum space of all maps $S \to \mathbb{S}$ can be endowed with structure of a quantum semigroup $\mathbb{H}$.
- If $\mathbb{S}$ is a quantum group then so is $\mathbb{H}$.
- If $\mathbb{H}$ is a quantum group then so is $\mathbb{S}$.
- Weird things happen when S is taken to be a quantum space.
- For more see: P.M. Sołtan ”On quantum maps into quantum semigroups”, to appear in *Houston Journal of Mathematics*.
- 
- THANK YOU!