Quantum group actions on discrete quantum spaces and quantum Clifford theory

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   - Actions on $M_n(\mathbb{C}) \oplus \widetilde{\mathsf{N}}$

- $\mathsf{M}=\prod\limits_{i\in\mathcal{I}}\mathsf{M}_i$ — product of von Neumann algebras,
- $\mathbb{G}$ — compact quantum group,
- $\alpha:\mathsf{M}\longrightarrow\mathsf{M}\bar{\otimes}L^\infty(\mathbb{G})$ — action of $\mathbb{G}$ on M:
    - α is an injective, unital, normal $*$-homomorphism,
    - $(\alpha\otimes\mathrm{id})\circ\alpha=(\mathrm{id}\otimes\Delta_{\mathbb{G}})\circ\alpha$.
- $\boldsymbol{p}_i:\mathsf{M}\longrightarrow\mathsf{M}_i$ — canonical projection.

Definition

We say that $i,j\in\mathcal{I}$ are α-**related** (writing $i\sim_\alpha j$) if

$$\exists\,x\in\mathsf{M}_i\quad(\boldsymbol{p}_j\otimes\mathrm{id})\alpha(x)\neq 0.$$

- Define $\alpha_{j,i} : \mathsf{M}_i \longrightarrow \mathsf{M}_j \bar{\otimes} L^\infty(\mathbb{G})$ by

$$\alpha_{j,i}(x) = (\boldsymbol{p}_j \otimes \mathrm{id})\alpha(x), \qquad x \in \mathsf{M}_i.$$

Then $\alpha_{j,i}$ is a normal $*$-homomorphism. Moreover for each $x \in \mathsf{M}_i$

$$\alpha(x) = \sum_{j \in \mathcal{I}} \alpha_{j,i}(x).$$

FACT

T.F.A.E. for $i, j \in \mathcal{I}$:

1. $i \sim_\alpha j$,
2. $\alpha_{j,i} \neq 0$,
3. $\alpha_{j,i}(\mathbb{1}_{\mathsf{M}_i}) \neq 0$.

- We say that α is **implemented** if there exist
  - a Hilbert space $\mathscr{H}$,
  - a faithful normal representation π of M on $\mathscr{H}$,
  - a unitary $U \in \mathrm{B}(\mathscr{H}) \bar{\otimes} L^{\infty}(\mathbb{G})$

  such that

$$(\pi \otimes \mathrm{id})\alpha(y) = U\big(\pi(y) \otimes \mathbb{1}\big)U^{*}, \qquad y \in \mathsf{M}.$$

- Any action can be implemented:

EXAMPLE

- π_0 — faithful representation of M on $\mathscr{H}_0$,
- $\mathrm{W}^{\mathbb{G}} \in \ell^{\infty}(\widehat{\mathbb{G}}) \bar{\otimes} L^{\infty}(\mathbb{G})$ — the Kac-Takesaki operator of $\mathbb{G}$,
- Define:
  - $\mathscr{H} = \mathscr{H}_0 \otimes L^2(\mathbb{G})$,
  - $\pi = (\pi_0 \otimes \mathrm{id})\circ\alpha : \mathsf{M} \longrightarrow \mathrm{B}(\mathscr{H}_0) \bar{\otimes} L^{\infty}(\mathbb{G}) \subset \mathrm{B}(\mathscr{H})$,
  - $U = {\mathrm{W}^{\mathbb{G}}}_{23} \in \mathrm{B}(\mathscr{H}_0) \bar{\otimes} \ell^{\infty}(\widehat{\mathbb{G}}) \bar{\otimes} L^{\infty}(\mathbb{G}) \subset \mathrm{B}(\mathscr{H}) \bar{\otimes} L^{\infty}(\mathbb{G})$.

- Assume α is implemented ($\mathscr{H}, \pi, U$ as before).
- We can assume further that $U \in \mathrm{B}(\mathscr{H}) \bar{\otimes} L^\infty(\mathbb{G})$ is a representation of $\mathbb{G}$:

$$(\mathrm{id} \otimes \Delta_{\mathbb{G}})U = U_{12}U_{13}.$$

- For each i let p_i be the unit of M_i (as a projection in M)
- Define $\mathscr{H}_i = \pi(p_i)\mathscr{H}$. We have

$$\mathscr{H} = \bigoplus_{i \in \mathcal{I}} \mathscr{H}_i \quad \text{and} \quad \pi(y) = \bigoplus_{i \in \mathcal{I}} \pi_i\big(\boldsymbol{p}_i(y)\big),$$

  where $\pi_i : \mathsf{M}_i \longrightarrow \mathrm{B}(\mathscr{H}_i)$ is a faithful representation.
- For $k, l \in \mathcal{I}$ let $U_{k,l} = \big(\pi(p_k) \otimes \mathbb{1}\big)U\big(\pi(p_l) \otimes \mathbb{1}\big)$.
- Implementation of α by U means:

$$(\pi_j \otimes \mathrm{id})\alpha_{j,i}(x) = U_{j,i}\big(\pi_i(x) \otimes \mathbb{1}\big){U_{j,i}}^* \qquad i, j \in \mathcal{I},\ x \in \mathsf{M}_i.$$

- Since $(\pi_j \otimes \mathrm{id})\big(\alpha_{j,i}(\mathbb{1}_{\mathsf{M}_i})\big) = U_{j,i}U_{j,i}{}^*$, for all $i,j \in \mathcal{I}$ we have

$$\big(\, i \sim_\alpha j \,\big) \iff \big(\, U_{j,i} \neq 0 \,\big).$$

Proposition

The relation $\sim_\alpha$ is symmetric.

Proof.

Assume α is implemented by a representation. We have $i \sim_\alpha j$ iff $U_{j,i} \neq 0$ iff there are $\xi \in \mathscr{H}_j$ and $\eta \in \mathscr{H}_i$ such that $(\omega_{\xi,\eta} \otimes \mathrm{id})U \neq 0$. Now $(\omega_{\xi,\eta} \otimes \mathrm{id})U \in \mathrm{D}(S)$ (domain of antipode) and

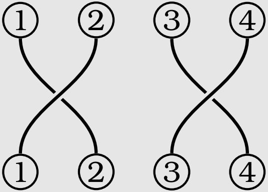
$$0 \neq S\big((\omega_{\xi,\eta} \otimes \mathrm{id})U\big) = (\omega_{\xi,\eta} \otimes \mathrm{id})(U^*).$$

This means that $0 \neq (U^*)_{j,i} = U_{i,j}{}^*$, so $U_{i,j} \neq 0$ and $i \sim_\alpha j$. ☐

Remark

The relation $\sim_\alpha$ need not be transitive. For example take

- $\mathsf{M} = L^\infty(\{1,2,3,4\}) = L^\infty(\{1\}) \oplus L^\infty(\{2,3\}) \oplus L^\infty(\{4\})$,
- $\mathbb{G} = \mathbb{Z}_2$ acting by



- Then $\{1\} \sim_\alpha \{2,3\}$ and $\{2,3\} \sim_\alpha \{4\}$, but $\{1\} \not\sim_\alpha \{4\}$.

PROPOSITION

If M_i is a factor for each i then $\sim_\alpha$ is an equivalence relation.

PROOF.

From $(\mathrm{id}\otimes\Delta_{\mathbb{G}})\circ\alpha=(\alpha\otimes\mathrm{id})\circ\alpha$ it follows that

$$(\mathrm{id}\otimes\Delta_{\mathbb{G}})\big(\alpha_{j,i}(x)\big)=\sum_{k\in\mathcal{I}}(\alpha_{j,k}\otimes\mathrm{id})\big(\alpha_{k,i}(x)\big),\qquad i,j\in\mathcal{I},\ x\in\mathsf{M}_i.$$

Since each M_i is a factor, for any $a,b\in\mathcal{I}$ we have $a\sim_\alpha b$ iff $\ker\alpha_{a,b}=\{0\}$. Assume $i\sim_\alpha l$ and $l\sim_\alpha j$. We have

$$(\mathrm{id}\otimes\Delta_{\mathbb{G}})\big(\alpha_{j,i}(\mathbb{1}_{\mathsf{M}_i})\big)=\sum_{k\in\mathcal{I}}(\alpha_{j,k}\otimes\mathrm{id})\big(\alpha_{k,i}(\mathbb{1}_{\mathsf{M}_i})\big)$$

(sum of orthogonal projections).

PROOF CONT'D.

- Since $i \sim_\alpha l$, we have $\alpha_{l,i}(\mathbb{1}_{\mathsf{M}_i}) \neq 0$,
- since $j \sim_\alpha l$, we have $\ker \alpha_{j,l} = \{0\}$.

It follows that

$$\begin{aligned}(\mathrm{id} \otimes \Delta_{\mathbb{G}})\bigl(\alpha_{j,i}(\mathbb{1}_{\mathsf{M}_i})\bigr) &= \sum_{k \in \mathcal{I}} (\alpha_{j,k} \otimes \mathrm{id})\bigl(\alpha_{k,i}(\mathbb{1}_{\mathsf{M}_i})\bigr) \\ &\geq (\alpha_{j,l} \otimes \mathrm{id})\bigl(\alpha_{l,i}(\mathbb{1}_{\mathsf{M}_i})\bigr) \neq 0,\end{aligned}$$

so $\alpha_{j,i}(\mathbb{1}_{\mathsf{M}_i}) \neq 0$, i.e. $i \sim_\alpha j$.
Finally, for any i there is j such that $i \sim_\alpha j$ (otherwise α would not be injective). Thus by symmetry and transitivity we get reflexivity of $\sim_\alpha$. □

DEFINITION

The classes of $\sim_\alpha$ will be called **orbits** of α.

- Let $A \subset \mathcal{I}$ be an equivalence class of $\sim_\alpha$. Then α restricts to an action on $\prod_{i \in A} \mathsf{M}_i$.
- Moreover, the projection $p_A = \sum_{i \in A} p_i$ is invariant:

$$\alpha(p_A) = p_A \otimes \mathbb{1}.$$

COROLLARY 1

If α is ergodic then $\sim_\alpha$ is the total relation.

COROLLARY 2

If $\mathsf{M}_i = M_{n_i}(\mathbb{C})$ for all $i \in \mathcal{I}$ then all orbits of α are finite.

THEOREM

Let α be an ergodic action of a compact quantum group $\mathbb{G}$ on a von Neumann algebra N of the form $\mathsf{N} = M_n(\mathbb{C}) \oplus \widetilde{\mathsf{N}}$. Then $\dim \mathsf{N} < +\infty$.

Sketch of proof of Corollary 2.

- Restrict to one class: $\forall\, i,j \quad i \sim_\alpha j$,
- take a minimal projection p in $\mathsf{M}^\alpha = \{m \in \mathsf{M} \,|\, \alpha(m) = m \otimes \mathbb{1}\}$,
- α restricts to an action on $p\mathsf{M}p$ which is ergodic,
- $p\mathsf{M}p$ is itself a product of matrix algebras, so by Theorem $\dim p\mathsf{M}p < +\infty$,
- thus $\mathcal{I}_p = \{i \in \mathcal{I} \,|\, p_i p \neq 0\}$ is finite,
- take $i \in \mathcal{I}_p$ and $j \in \mathcal{I} \setminus \mathcal{I}_p$. We have $p_i p \neq 0$, and

$$\alpha_{j,i}(p_i p) = (\boldsymbol{p}_j \otimes \mathrm{id})\big(\alpha(p_i p)\big) \leq (\boldsymbol{p}_j \otimes \mathrm{id})\big(\alpha(p)\big) = \boldsymbol{p}_j(p) \otimes \mathbb{1} = 0.$$

But $i \sim_\alpha j$, so $\ker \alpha_{j,i} = \{0\}$ — a contradiction.
- Thus $\mathcal{I} = \mathcal{I}_p$ is finite.

□

- Let $\mathbb{\Gamma}$ be a discrete quantum group:

$$\ell^{\infty}(\mathbb{\Gamma}) = \prod_{\gamma \in \operatorname{Irr}\widehat{\mathbb{\Gamma}}} M_{n_\gamma}(\mathbb{C}),$$

- and let $\mathbb{\Lambda}$ be a quantum subgroup of $\mathbb{\Gamma}$:
  - $L^{\infty}(\widehat{\mathbb{\Lambda}}) \subset L^{\infty}(\widehat{\mathbb{\Gamma}})$, ($\mathbb{\Lambda}$ is closed)
  - $\boldsymbol{\pi} : \ell^{\infty}(\mathbb{\Gamma}) \longrightarrow \ell^{\infty}(\mathbb{\Lambda})$. ($\mathbb{\Lambda}$ is open)
- Put $\ell^{\infty}(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) = \left\{x \in \ell^{\infty}(\mathbb{\Gamma}) \,\middle|\, (\boldsymbol{\pi} \otimes \mathrm{id})\Delta_{\mathbb{\Gamma}}(x) = \mathbb{1} \otimes x\right\}$.
- Let $\mathbb{G} = \widehat{\mathbb{\Gamma}}$. We have

$$\mathrm{W}^{\mathbb{G}}\big(\ell^{\infty}(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \otimes \mathbb{1}\big)\mathrm{W}^{\mathbb{G}*} \subset \ell^{\infty}(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \,\bar{\otimes}\, L^{\infty}(\widehat{\mathbb{\Gamma}})$$

which yields an action of $\mathbb{G}$ on $\ell^{\infty}(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$:

$$\alpha(x) = \mathrm{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{G}*}, \qquad x \in \ell^{\infty}(\mathbb{\Lambda}\backslash\mathbb{\Gamma}).$$

Example

Consider a special case:

- let $\mathbb{H} \subset \mathbb{G}$ be a normal closed quantum subgroup,
- let $\mathbb{\Gamma} = \widehat{\mathbb{G}}$ and $\mathbb{\Lambda} = \widehat{\mathbb{G}/\mathbb{H}}$.

Then $\mathbb{G}$ acts on $\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) = \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) = \ell^\infty(\widehat{\mathbb{H}})$.

- $\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ is a product of matrix algebras:

$$\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) = \prod_{i \in \mathcal{I}} \mathsf{M}_i$$

  with each $\mathsf{M}_i = M_{m_i}(\mathbb{C})$.
- The action of $\mathbb{G} = \widehat{\mathbb{\Gamma}}$ on $\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ defines the equivalence relation $\sim_\alpha$ on $\mathcal{I}$.

- Representations of $\mathbb{G}=\widehat{\mathbb{\Gamma}}$ are all of the form

$$U=(\varphi\otimes\mathrm{id})\mathrm{W}^{\mathbb{G}},$$

where φ is a representation of $\ell^\infty(\mathbb{\Gamma})$.
- Irreps correspond to matrix blocks in the decomposition

$$\ell^\infty(\mathbb{\Gamma})=\prod_{\gamma}M_{n_\gamma}(\mathbb{C}).$$

- Since $\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})\subset\ell^\infty(\mathbb{\Gamma})$, representations of $\ell^\infty(\mathbb{\Gamma})$ can be restricted to $\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$.
- When $\mathbb{\Lambda}=\widehat{\mathbb{G}/\mathbb{H}}$ for a normal $\mathbb{H}\subset\mathbb{G}$ this is exactly restricting representations of $\mathbb{G}$ to $\mathbb{H}$ ($\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})=\ell^\infty(\widehat{\mathbb{H}})$).
- Classical theorem of Clifford says that an irrep of G restricted to a normal $H\subset G$ is equivalent to a direct sum of irreps of H forming precisely one orbit of the action of G on $\operatorname{Irr}H$ by conjugation.

- Denote by $\mathbb{1}_j$ the unit of $\mathsf{M}_j \subset \ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ viewed as a projection in $\ell^\infty(\mathbb{\Gamma})$.

Theorem

For any $i \in I$ the element

$$\sum_{j\sim_\alpha i} \mathbb{1}_j \in \ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \subset \ell^\infty(\mathbb{\Gamma})$$

is the central support $z(\mathbb{1}_i)$ in $\ell^\infty(\mathbb{\Gamma})$ of the projection $\mathbb{1}_i$. Moreover $z(\mathbb{1}_i)$ is orthogonal to $z(\mathbb{1}_j)$ if i is not equivalent to j (i.e. i and j are not in the same orbit).

- In particular for any $\kappa \in \operatorname{Irr}\widehat{\mathbb{\Gamma}}$ there exists $i \in \mathcal{I}$ such that
  1. for all $j \in \mathcal{I}$ we have $p_\kappa \mathbb{1}_j \neq 0$ if and only if $j \sim_\alpha i$,
  2. we have $p_\kappa\left(\sum_{j\sim_\alpha i} \mathbb{1}_j\right) = p_\kappa$.

Example revisited

- When $\Lambda = \widehat{\mathbb{G}/\mathbb{H}}$ for a closed normal subgroup $\mathbb{H}$ of $\mathbb{G}$, the theorem says that for an irrep κ of $\mathbb{G}$ (or $\ell^\infty(\widehat{\mathbb{G}})$) the restriction of κ to $\mathbb{H}$ (or $\ell^\infty(\widehat{\mathbb{H}})$) is a direct sum of irreps of $\mathbb{H}$ constituting one class of the equivalence relation $\sim_\alpha$ on $\mathcal{I} = \operatorname{Irr}\mathbb{H}$.
- For classical groups G and H the irreps of H in one orbit of the action of G (by conjugation) all have the same dimension.

Theorem

Let $\mathbb{G}$ be a compact quantum group of Kac type and let $\mathbb{H}$ be a closed normal quantum subgroup of $\mathbb{G}$. Then any two irreducible representations σ and τ of $\mathbb{H}$ in the same orbit have the same dimension. Moreover, if π is any irreducible representation of $\mathbb{G}$ with $\pi(\mathbb{1}_\sigma) \neq 0$, then also the multiplicity of σ in π is the same as the multiplicity of τ in π.

THEOREM

Consider the following three conditions

1. $\widehat{\mathbb{G}}$ *is* ***torsion free***,
2. $\mathbb{G}$ *is satisfies the* (TO)***-condition***,
3. $\mathbb{G}$ *is* ***connected***.

All actions of $\mathbb{G}$ *on finite dimensional* C^**-algebras are direct sums of actions Morita equivalent to trivial action on* $\mathbb{C}$

For any action of $\mathbb{G}$ *on a product of matrix algebras the orbits are trivial*

There is no finite quantum group $\mathbb{H}$ *such that* $\mathrm{Pol}(\mathbb{H}) \subset \mathrm{Pol}(\mathbb{G})$ *as a Hopf* $*$*-subalgebra*

Then

$$① \implies ② \implies ③.$$

In general neither of the implications can be reversed.

Definition

Let $\mathbb{\Lambda}$ be a quantum subgroup of a discrete quantum group $\mathbb{\Gamma}$. For $\sigma, \tau \in \operatorname{Irr}\widehat{\mathbb{\Gamma}}$ we say that σ and τ are $\mathbb{\Lambda}$-**related** if there exists $\gamma \in \operatorname{Irr}\widehat{\mathbb{\Lambda}}$ such that $\tau \subset \sigma \text{⫪} \gamma$.

- Recall that in this situation we have an action α of $\mathbb{G}$ on $\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) = \prod_{i\in\mathcal{I}} \mathsf{M}_i$.
- For $i \in \mathcal{I}$ define $\mathbb{\Gamma}\text{-supp}(\mathbb{1}_i) = \left\{\kappa \in \operatorname{Irr}\widehat{\mathbb{\Gamma}} \,\middle|\, p_\kappa \mathbb{1}_i \neq 0\right\}$.

Theorem

1. *For $i, j \in \mathcal{I}$ we have $i \sim_\alpha j$ iff $\mathbb{\Gamma}\text{-supp}(\mathbb{1}_i) = \mathbb{\Gamma}\text{-supp}(\mathbb{1}_j)$,*
2. *two elements $\sigma, \tau \in \operatorname{Irr}\widehat{\mathbb{\Gamma}}$ are $\mathbb{\Lambda}$-related iff there exists $i \in \mathcal{I}$ such that $\sigma, \tau \in \mathbb{\Gamma}\text{-supp}(\mathbb{1}_i)$.*

- Let $\alpha : \mathsf{N} \to \mathsf{N} \bar{\otimes} L^\infty(\mathbb{G})$ be an **ergodic** action,
- assume $N = M_n(\mathbb{C}) \oplus \widetilde{\mathsf{N}}$.
- There is a unique invariant state φ on N, let $L^2(\mathsf{N})$ be the associated G.N.S. space.
- Define $\mathcal{G} : L^2(\mathsf{N}) \otimes L^2(\mathsf{N}) \longrightarrow L^2(\mathsf{N}) \otimes L^2(\mathbb{G})$ by extending

$$x \otimes y \longmapsto \alpha(y)(x \otimes \mathbb{1}), \qquad x, y \in \mathrm{Pol}(\mathsf{N}),$$

  where $\mathrm{Pol}(\mathsf{N})$ is the **algebraic core** (or **Podleś subalgebra** or **polynomial subalgebra**) of N.
- One can show that $\mathcal{G} \in \mathsf{N} \otimes \mathrm{B}(L^2(\mathsf{N}), L^2(\mathbb{G}))$, so for $\omega \in \mathsf{N}_*$ we have

$$L_\omega = (\omega \otimes \mathrm{id})(\mathcal{G}^*) \in \mathrm{B}(L^2(\mathbb{G}), L^2(\mathsf{N})).$$

- Put

$$\mathrm{c}_0(\widehat{\mathsf{N}}) = \overline{\left\{ L_\omega \,\middle|\, \omega \in \mathsf{N}_* \right\}}$$

- $c_0(\widehat{\mathsf{N}})$ can be shown to be a Hilbert C*-module over $c_0(\widehat{\mathbb{G}})$ of the form
$$c_0(\widehat{\mathsf{N}}) = \bigoplus_{\kappa \in \operatorname{Irr}\mathbb{G}} M_{m_\kappa, n_\kappa}(\mathbb{C})$$
for some non-negative integers $\{m_\kappa\}_{\kappa \in \operatorname{Irr}\mathbb{G}}$ (c_0-direct sum).
- Finiteness of each m_κ follows from ergodicity of α.
- In fact the summand $M_{m_\kappa, n_\kappa}(\mathbb{C})$ is
$$\left\{L_{\varphi(\cdot\, x)} \,\middle|\, x \in \mathsf{N} \text{ transforms according to } \kappa\right\}$$
($x \in \operatorname{Pol}(\mathsf{N})^\kappa$).
- If N is infinite dimensional then so is $c_0(\widehat{\mathsf{N}})$.

- Let $\eta : \mathsf{N} = M_n(\mathbb{C}) \oplus \widetilde{\mathsf{N}} \longrightarrow M_n(\mathbb{C})$ be the canonical projection.
- Define $\eta_{i,j} \in \mathsf{N}_*$ by

$$\eta(x) = \sum_{i,j} \eta_{i,j}(x) e_{i,j}, \qquad x \in \mathsf{N},$$

- and let $L_\eta = \sum_{i,j} e_{i,j} \otimes L_{\eta_{i,j}} \in M_n(\mathbb{C}) \otimes c_0(\widehat{\mathsf{N}})$.
- If $\dim \mathsf{N} = +\infty$ there are infinitely many different κ with $L_{\varphi(\cdot\, x_\kappa)} p_\kappa = L_{\varphi(\cdot\, x_\kappa)}$ for some non-zero $x_\kappa \in \mathrm{Pol}(\mathsf{N})^\kappa$.
- Then, via somewhat complicated calculations, we get

$$\|L_{\varphi(\cdot\, x_\kappa)}\| = \left\|(\mathbb{1} \otimes p_\kappa) {L_\eta}^* (\mathbb{1} \otimes L_{\varphi(\cdot\, x_\kappa)})\right\| \leq \left\|L_\eta(\mathbb{1} \otimes p_\kappa)\right\| \|L_{\varphi(\cdot\, x_\kappa)}\|,$$

  so $\left\|L_\eta(\mathbb{1} \otimes p_\kappa)\right\| \geq 1$ for infinitely many different κ.
- This is a contradiction with $L_\eta \in M_n(\mathbb{C}) \otimes c_0(\widehat{\mathsf{N}})$.

Thank you for your attention.

☺