COMPACT QUANTUM GROUP ACTIONS ON DISCRETE QUANTUM SPACES AND QUANTUM CLIFFORD THEORY

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$$M = \prod_{i \in I} M_i$$

$p_i : M \to M_i$ CANONICAL PROJECTION, p_i = UNIT OF M_i INSIDE M

$\alpha : M \to M \bar{\otimes} L^\infty(\mathbb{G})$ - ACTION OF $\mathbb{G}$ ON M

$$(\alpha \otimes id) \circ \alpha = (id \otimes \Delta_{\mathbb{G}}) \circ \alpha$$

(α - INJECTIVE, UNITAL, NORMAL $*$-HOMOMORPHISM)

<u>DEF.</u> $i, j \in I$

$$i \sim_\alpha j \overset{DEF}{\iff} \exists\, x \in M_i \quad (p_j \otimes id)\,\alpha(x) \neq 0$$

- $\alpha_{ji} : M_i \to M_j \bar{\otimes} L^\infty(\mathbb{G})$, $\alpha_{ji}(x) := (p_j \otimes id)\,\alpha(x)$, $x \in M_i$

CLEARLY FOR $x \in M_i$ $\quad \alpha(x) = \sum'_{j \in I} \alpha_{ji}(x)$

T.F.A.E.

(1) $i \sim_\alpha j$

(2) $\alpha_{ij} \neq 0$

(3) $\alpha_{ji}(\mathbb{1}_{M_i}) \neq 0$

$U \in B(H) \bar{\otimes} L^\infty(\mathbb{G})$
UNITARY
$(id \otimes \Delta)\, U = U_{12} U_{13}$

- α IS <u>IMPLEMENTED</u> IF $\exists\, H, \pi, U$

HILBERT SPACE

REP. OF $\mathbb{G}$ ON H

FAITHFUL REP. OF M ON H

$$(\pi \otimes id)\,\alpha(y) = U(\pi(y) \otimes \mathbb{1})\,U^*, \qquad y \in M$$

EXAMPLE: (H_0, π_0) - FAITHFUL REP. OF M

$$H := H_0 \otimes L^2(\mathbb{G})$$

$$\pi := (\pi_0 \otimes id) \circ \alpha \; : \; M \longrightarrow B(H_0) \bar{\otimes} L^\infty(\mathbb{G}) \subset B(H)$$

$$U := W^{\mathbb{G}}_{23} \in B(H_0) \bar{\otimes} \ell^\infty(\hat{\mathbb{G}}) \bar{\otimes} L^\infty(\mathbb{G}) \subset B(H) \bar{\otimes} L^\infty(\mathbb{G})$$

FOR $y \in M$

$$(\pi \otimes id)\alpha(y) = (\pi_0 \otimes id \otimes id)(\alpha \otimes id)\alpha(y) = (\pi_0 \otimes id \otimes id)(id \otimes \Delta_{\mathbb{G}})\alpha(y)$$

$$= (id \otimes \Delta)(\pi_0 \otimes id)\alpha(y) = (id \otimes \Delta)\pi(y)$$

$$= W_{23}(\pi(y) \otimes 1)W_{23}^* = U(\pi(y) \otimes 1)U^*$$

$\longrightarrow$ $H_i := \pi(p_i)H$, $\quad H = \bigoplus_{i \in I} H_i$, $\quad \pi(y) = \bigoplus_{i \in I} \pi_i(y_i)$, $\quad y \in M$

$\qquad y_i = p_i(y)$

$\longrightarrow$ $U_{kl} := (\pi(p_k) \otimes 1)\, U\, (\pi(p_l) \otimes 1)$, $\quad k, l \in I$

$\longrightarrow$ U IMPLEMENTS α:

$$(\pi_j \otimes id)\alpha_{ji}(x) = U_{ji}(\pi_i(x) \otimes 1)U_{ji}^*, \qquad i,j \in I, \quad x \in M_i$$

- WE HAVE FOR ALL $i,j \in I$

$$(\pi_j \otimes id)(\alpha_{ji}(1_{M_i})) = U_{ji}U_{ji}^*$$

SO

$$(i \sim_\alpha j) \Longleftrightarrow (U_{ji} \neq 0).$$

PROP. THE RELATION $\sim_\alpha$ IS SYMMETRIC.

ASSUME α IS IMPLEMENTED BY $U \in B(H) \otimes L^\infty(\mathbb{G})$.

$$i \sim_\alpha j \iff U_{ji} \neq 0 \iff \exists\, \xi \in H_j,\ \eta \in H_i \quad (\omega_{\xi\eta} \otimes id)U \neq 0$$

NOW $(\omega_{\xi\eta} \otimes id) U \in D(S)$ AND

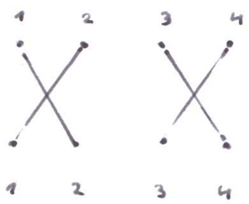
$$0 \neq S\big((\omega_{\xi\eta} \otimes id) U\big) = (\omega_{\xi\eta} \otimes id) U^*$$

THIS MEANS THAT $(U^*)_{ji} = U_{ij}^* \neq 0$ SO $U_{ij} \neq 0$ AND $j \sim_\alpha i$.

- THE RELATION $\sim_\alpha$ NEED NOT BE TRANSITIVE:

$\mathbb{G} = \mathbb{Z}_2$ ACTING ON $L^\infty(\{1,2,3,4\}) = L^\infty(\{1\}) \oplus L^\infty(\{2,3\}) \oplus L^\infty(\{4\})$

BY



CLEARLY $\{1\} \sim \{2,3\}$ & $\{2,3\} \sim \{4\}$, BUT $\{1\} \not\sim \{4\}$.

PROP. IF FOR EACH $i \in I$ M_i IS A FACTOR THEN $\sim_\alpha$ IS AN EQUIVALENCE RELATION

FROM $(\alpha \otimes id) \circ \alpha = (id \otimes \Delta_{\mathbb{G}}) \circ \alpha$ IT FOLLOWS THAT

$$(id \otimes \Delta_{\mathbb{G}})(\alpha_{ji}(x)) = \sum_{k \in I} (\alpha_{jk} \otimes id)(\alpha_{ki}(x)), \qquad i,j \in I,\ x \in M_i$$

MOREOVER, SINCE EACH M_i IS A FACTOR, WE HAVE

$$a \sim_\alpha b \iff \ker \alpha_{ab} = \{0\}.$$

ASSUME $i \sim_\alpha l$ AND $j \sim_\alpha l$. THEN

$$(\mathrm{id} \otimes \Delta_{\mathbb{G}})(\alpha_{ji}(\mathbb{1}_{M_i})) = \underbrace{\sum_{k \in I}}_{\text{ORTHOGONAL SUM OF PROJECTIONS}} (\alpha_{jk} \otimes \mathrm{id})(\alpha_{ki}(\mathbb{1}_{M_i})) \geq (\alpha_{jl} \otimes \mathrm{id})\,\alpha_{li}(\mathbb{1}_{M_i}) \neq 0$$

SO $\alpha_{ji}(\mathbb{1}_{M_i}) \neq 0$ AND CONSEQUENTLY $i \sim_\alpha j$.

NOW FOR $i \in I$ THERE IS $j \in I$ SUCH THAT $\alpha_{ji}(\mathbb{1}_{M_i}) \neq 0$ (OTHERWISE $\ker \alpha \neq \{0\}$) SO $i \sim_\alpha j$. BY SYMMETRY & TRANSITIVITY $\sim_\alpha$ IS ALSO REFLEXIVE.

→ LET $A \subset I$ BE AN EQUIVALENCE CLASS OF $\sim_\alpha$. THEN $\alpha : M \longrightarrow M \bar{\otimes} L^\infty(\mathbb{G})$ RESTRICTS TO $\prod_{i \in A} M_i$ GIVING AN ACTION OF $\mathbb{G}$ ON $\prod_{i \in A} M_i$. IN PARTICULAR

$$\alpha(\mathbb{1}_A) = \mathbb{1}_A \otimes \mathbb{1}$$

$$\mathbb{1}_A = \sum_{i \in A} p_i$$

<u>COR.</u> IF α IS ERGODIC, $\sim_\alpha$ IS THE TOTAL RELATION.

THM. LET α BE AN ERGODIC ACTION OF A COMPACT QUANTUM GROUP ON A VON NEUMANN ALGEBRA N. ASSUME THAT $N = M_n(\mathbb{C}) \oplus \tilde{N}$. THEN $\dim N < +\infty$.

COR. LET α BE AN ACTION OF A COMPACT QUANTUM GROUP ON $\prod_{i\in I} M_{n_i}(\mathbb{C})$. THEN ALL ORBITS OF α (EQUIVALENCE CLASSES OF $\sim_\alpha$) ARE FINITE

→ RESTRICT TO ONE CLASS : ASSUME $\forall_{i,j}\ i \sim_\alpha j$

→ TAKE p - MINIMAL PROJECTION IN $\{ m \in M \mid \alpha(m) = m \otimes \mathbb{1} \}$

→ α RESTRICTS TO AN ACTION ON pMp WHICH IS ERGODIC

→ pMp IS ITSELF A PRODUCT OF MATRIX ALGEBRAS SO BY THM. $\dim pMp < +\infty$.

→ THUS $I_p = \{ i \in I \mid p_i p \neq 0 \}$ IS FINITE $(p_i p = p p_i p)$

→ TAKE $i \in I_p$, $j \in I \setminus I_p$. $p_i p \neq 0$, BUT

$$\alpha_{ji}(p_i p) = (p_j \otimes id)(\alpha(p_i p)) \leq (p_j \otimes id)(\alpha(p)) = \underbrace{p_j(p)}_{=0\ (p_j p = 0)} \otimes \mathbb{1} = 0$$

IF $i \sim_\alpha j$ THEN $\alpha_{ji} \neq 0$ i.e. $\ker \alpha_{ji} = \{0\}$.

BUT $p_i p_j \neq 0$, SO WE ARRIVE AT A CONTRADICTION.

Γ - DISCRETE QUANTUM GROUP $\quad \ell^\infty(\Gamma) = \prod_{\gamma \in \mathrm{Irr}\,\hat{\Gamma}} M_{n_\gamma}(\mathbb{C})$

$$\Delta_\Gamma : \ell^\infty(\Gamma) \to \ell^\infty(\Gamma) \bar{\otimes} \ell^\infty(\Gamma)$$

Λ - QUANTUM SUBGROUP OF Γ (ALSO DISCRETE)

- $L^\infty(\hat{\Lambda}) \subset L^\infty(\hat{\Gamma})$ (Λ IS CLOSED)
- ALSO $\pi : \ell^\infty(\Gamma) \twoheadrightarrow \ell^\infty(\Lambda)$ (Λ IS OPEN)

$\to \ell^\infty(\Lambda \backslash \Gamma) = \{ x \in \ell^\infty(\Gamma) \mid (\pi \otimes \mathrm{id})\Delta_\Gamma(x) = \mathbb{1} \otimes x \}$

$\to \ell^\infty(\Lambda \backslash \Gamma)$ IS A PRODUCT OF MATRIX ALGEBRAS

$\to$ WE HAVE

$$W^{\hat{\Gamma}} (\ell^\infty(\Lambda\backslash\Gamma) \otimes \mathbb{1}) W^{\hat{\Gamma}*} \subset \ell^\infty(\Lambda\backslash\Gamma) \bar{\otimes} L^\infty(\hat{\Gamma})$$

WHICH YIELDS AN ACTION OF $\hat{\Gamma}$ ON $\Lambda \backslash \Gamma$

- EXAMPLE: $\mathbb{G}$ - COMPACT QUANTUM GROUP

  $\mathbb{H}$ - CLOSED NORMAL QUANTUM SUBGROUP OF $\mathbb{G}$

  $\Gamma := \hat{\mathbb{G}}, \quad \Lambda := \widehat{\mathbb{G}/\mathbb{H}}$

  IN THIS CASE Λ IS NORMAL AND

  $$\Lambda \backslash \Gamma = \Gamma / \Lambda = \hat{\mathbb{H}}$$

  $(\ell^\infty(\hat{\mathbb{H}}) \subset \ell^\infty(\hat{\mathbb{G}}))$

$\to$ WITH $\alpha : \ell^\infty(\Lambda\backslash\Gamma) \to \ell^\infty(\Lambda\backslash\Gamma) \bar{\otimes} L^\infty(\mathbb{G})$

$\alpha(x) = W^{\hat{\Gamma}}(x \otimes \mathbb{1}) W^{\hat{\Gamma}*}$ AND $\ell^\infty(\Lambda\backslash\Gamma) = \prod_{i \in I} M_i$

$(M_i = M_{m_i}(\mathbb{C})$ FOR EACH $i)$ WE GET

$\sim_\alpha$ ON I.

THM. FOR ANY $\varkappa \in \mathrm{Irr}\,\hat{\Gamma}$ THERE EXISTS $i \in I$ SUCH THAT

(1) FOR $j \in I$ WE HAVE $p_\varkappa \mathbb{1}_{M_j} \neq 0$ IFF $j \sim_\alpha i$

(2) WE HAVE $p_\varkappa \Big(\sum\limits_{j \sim_\alpha i} \mathbb{1}_{M_j} \Big) = p_\varkappa$

MOREOVER, FOR ANY $i \in I$ THE ELEMENT $\sum\limits_{j \sim_\alpha i} \mathbb{1}_{M_j} \in \ell^\infty(\Lambda \backslash \Gamma) \subset \ell^\infty(\Gamma)$ IS THE CENTRAL SUPPORT IN $\ell^\infty(\Gamma)$ OF $\mathbb{1}_{M_i}$.

IN THE PROOF WE SHOW (AMONG OTHER THINGS) THAT FOR $i,j \in I$

$$i \sim_\alpha j \iff \mathrm{supp}_\Gamma \mathbb{1}_{M_i} = \mathrm{supp}_\Gamma \mathbb{1}_{M_j}$$

WHERE $\mathrm{supp}_\Gamma \mathbb{1}_{M_k} = \{ \gamma \in \mathrm{Irr}\,\hat{\Gamma} \mid p_\gamma \mathbb{1}_{M_k} \neq 0 \}$

APPLYING THE THEOREM TO $\Gamma = \hat{\mathbb{G}}$, $\Lambda = \widehat{\mathbb{G}/\mathbb{H}}$ FOR $\mathbb{G}$ - COMPACT QUANTUM GROUP, $\mathbb{H}$ - CLOSED NORMAL Q. SUBGROUP WE OBTAIN

$$\ell^\infty(\Lambda \backslash \Gamma) = \ell^\infty(\hat{\mathbb{H}}) = \prod_{\sigma \in \mathrm{Irr}\,\mathbb{H}} M_{n_\sigma}(\mathbb{C})$$

AND: FOR ANY $\pi \in \mathrm{Irr}\,\mathbb{G}$ THERE EXISTS $\sigma \in \mathrm{Irr}\,\mathbb{H}$ SUCH THAT

(1) FOR $\tau \in \mathrm{Irr}\,\mathbb{H}$ $\pi(p_\tau) \neq 0$ IFF $\tau \sim_\alpha \sigma$

(2) $\pi \Big(\sum\limits_{\varsigma \sim_\alpha \sigma} p_\varsigma \Big) = \mathbb{1}$

HERE $\alpha : \ell^\infty(\hat{\mathbb{H}}) \longrightarrow \ell^\infty(\hat{\mathbb{H}}) \bar{\otimes} L^\infty(\mathbb{G})$ IS THE ACTION BY INNER AUTOMORPHISMS

THIS ACTION RESTRICTS TO $\mathcal{Z}(\ell^\infty(\hat{\mathbb{H}})) = \ell^\infty(\mathrm{Irr}\,\mathbb{H})$ AND THE RESULTING EQUIVALENCE RELATIONS ARE EQUAL.

THM. ASSUME $\mathbb{G}$ IS OF KAC TYPE. THEN IF $\sigma_1, \sigma_2 \in \operatorname{Irr} \mathbb{H}$ AND $\sigma_1 \sim_\alpha \sigma_2$ THEN FOR ANY $\pi \in \operatorname{Irr} \mathbb{G}$

$$\dim \pi(p_{\sigma_1}) = \dim \pi(p_{\sigma_2})$$

THIS MEANS THAT IN RESTRICTION OF π TO $\mathbb{H}$ THE ISOTYPICAL COMPONENTS CORRESPONDING TO σ_1 AND σ_2 HAVE THE SAME DIMENSION.

- VERGNIOUX RELATION: GIVEN Γ & $\Lambda \subset \Gamma$ WE INTRODUCE AN EQUIVALENCE RELATION ON $\operatorname{Irr} \hat{\Gamma}$:

$$\sigma \sim_\Lambda \tau \iff \exists\, \gamma \in \operatorname{Irr} \hat{\Lambda} \quad \tau \subset \sigma \oplus \gamma$$

THM. LET $\ell^\infty(\Lambda \backslash \Gamma) = \prod_{i \in I} M_i$ WITH M_i - SIMPLE. THEN

$\sigma \sim_\Lambda \tau$ IFF THERE EXISTS $i \in I$ SUCH THAT $\sigma, \tau \in \operatorname{supp}_\Gamma \mathbb{1}_{M_i}$