On quantum groups from multiplicative unitaries

Piotr M. Sołtan

Joint work with S.L. Woronowicz

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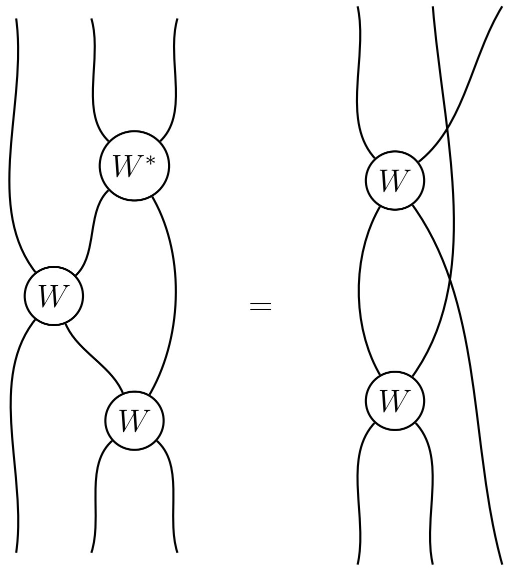


1 Modular multiplicative unitaries

$\mathcal{H}$ – (separable) Hilbert space

Modular multiplicative unitary: $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$

✏ W is unitary

✏ W is multiplicative: $W_{23}W_{12}W_{23}^{*} = W_{12}W_{13}$

✏ $\exists\ Q, \widehat{Q}$ - positive, self adjoint on $\mathcal{H}$ such that $W\left(\widehat{Q} \otimes Q\right)W^{*} = \widehat{Q} \otimes Q$

✏ $\exists$ unitary $\widetilde{W} \in \mathrm{B}\left(\overline{\mathcal{H}} \otimes \mathcal{H}\right)$ such that

$$(x \otimes u|W|z \otimes y) = \left(\overline{z} \otimes Qu\middle|\widetilde{W}\middle|\overline{x} \otimes Q^{-1}y\right)$$

2 Multiplicative unitaries give quantum groups

$W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ – modular multiplicative unitary

☛ $A = \left\{(\omega \otimes \mathrm{id})(W) \middle| \omega \in \mathrm{B}(\mathcal{H})_*\right\}^-$ is a C*-algebra

☛ For $a \in A$ formula $\Delta(a) = W(a \otimes I)W^*$ defines a coassociative $\Delta \in \mathrm{Mor}(A, A \otimes A)$

(important: $\mathrm{B}(\mathcal{H})_* = \mathcal{K}(\mathcal{H})^*$ – depends on $\mathcal{H}$)

Definition: $\mathbb{G} = (A, \Delta)$ – quantum group.

($\mathbb{G} = (A, \Delta)$ *is a* quantum group *if* A *and* Δ *are obtained from a modular multiplicative unitary as indicated above*)

?? Dependence on choice of m.m.u. ??

3 More structure

❏ $\exists!$ closed $\kappa : A \to A$ such that

☞ $\kappa\big((\omega \otimes \mathrm{id})(W)\big) = (\omega \otimes \mathrm{id})(W^*)$

☞ $a, b \in \mathrm{D}(\kappa) \Rightarrow ab \in \mathrm{D}(\kappa)$ and $\kappa(ab) = \kappa(b)\kappa(a)$

☞ $\kappa\big(\kappa(a)^*\big)^* = a$

❏ $\kappa = R \circ \tau_{\frac{i}{2}}$

✌ $(\tau_t)_{t\in\mathbb{R}}$ – one parameter group of automorphisms of A:

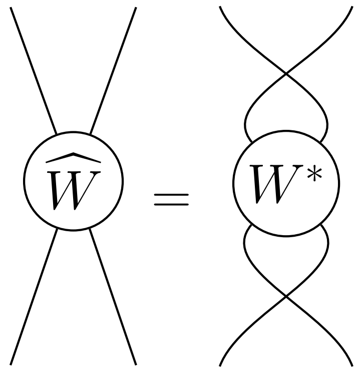
$$\tau_t(a) = Q^{2it} a Q^{-2it}$$

✌ R – ultraweakly continuous involutive antiautomorphism of A

❏ $\widehat{\mathbb{G}}$ – obtained as $\mathbb{G}$ from m.m.u. $\widehat{W}$ ($\widehat{\mathbb{G}}$ is called the reduced dual of $\mathbb{G}$)

4 Dual modular multiplicative unitary

$\widehat{W}$ given by



is also a modular multiplicative unitary

5 Main theorem

Let $\mathbb{G} = (A, \Delta)$ be a quantum group. Let $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ be a m.m.u. giving rise to $\mathbb{G}$. Then

- the ultraweak topology on $A \subset \mathrm{B}(\mathcal{H})$
- the unitary coinverse R
- the scaling group $(\tau_t)_{t \in \mathbb{R}}$
- the coinverse κ
- the reduced dual
- and many other important things

do not depend on the choice of $\mathcal{H}$ and W

6 Method of proof: step 1

❶ Develop theory of unitary representations of quantum groups

➠ direct sums

➠ tensor products

➠ contragradient representations

➠ intertwining operators

➠ equivalence and quasi-equivalence

➠ C*-algebras generated by representations

7 Method of proof: steps 2–7

❷ Notice that any m.m.u. giving rise to $\mathbb{G}$ is a representation and all such representations are **right absorbing**

❸ Notice that all right absorbing representations are quasi-equivalent

❹ Notice that quasi-equivalent representations generate isomorphicC*-algebras

❺ Notice that this isomorphism is a homeomorphism for the ultraweak topologies inherited from natural embeddings into operators

❻ Notice that A is a special case of C*-algebra generated by a representation

❼ Notice that we have already taken care of the ultraweak topology and deduce the other points of the theorem

8 One application

- There is a special – "maximal" representation $\mathbb{W} \in \mathrm{M}\big(\mathcal{K}(H_{\mathbb{W}}) \otimes A\big)$
- Representations of the C*-algebra $\widehat{A}_{\mathrm{u}}$ generated by $\mathbb{W}$ are in bijection with representations of $\mathbb{G}$
- $\widehat{A}_{\mathrm{u}}$ has comultiplication $\widehat{\Delta}_{\mathrm{u}}$ and all other structure $\widehat{\kappa}^{\mathrm{u}}$, $\widehat{\tau}^{\mathrm{u}}$ and $\widehat{R}^{\mathrm{u}}$
- $\big(\widehat{A}_{\mathrm{u}}, \widehat{\Delta}_{\mathrm{u}}\big)$ has a counit $\widehat{e}^{\mathrm{u}}$
- $\big(\widehat{A}_{\mathrm{u}}, \widehat{\Delta}_{\mathrm{u}}\big)$ is not a quantum group, but it is not far from being one
- If $\widehat{A}$ has a continuous counit then $\widehat{A} = \widehat{A}_{\mathrm{u}}$

9 What we are saying

- A quantum group is described by a C*-algebra with comultiplication – not a C*-algebra represented in $\mathrm{B}(\mathcal{H})$ with a comultiplication
- All important features coming from such a representation are determined uniquely by (A, Δ) – not by specific realization in Hilbert space
- We don't know how to prove existence of Haar measure, but we have shown existence of "Haar measure class" – this is, in effect, the ultraweak topology on A
- In particular this means that defining quantum groups via von Neumann algebras actually makes sense