Podleś spheres for the braided quantum $\mathrm{SU}(2)$

The Frontier of Quantum Dynamics
Banach Center, Warsaw

Piotr M. Sołtan

Department of Mathematical Methods in Physics
Faculty of Physics, University of Warsaw

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- We will consider the category $\mathfrak{C}^*_{\mathbb{T}}$ of C*-algebras endowed with an action of $\mathbb{T}$ and equivariant morphisms.
- Given $\mathsf{A} \in \mathrm{Ob}(\mathfrak{C}^*_{\mathbb{T}})$ the action will be described by
$$\rho^{\mathsf{A}} \in \mathrm{Mor}\big(\mathsf{A}, \mathsf{C}(\mathbb{T}) \otimes \mathsf{A}\big)$$
such that $(\mathbf{id} \otimes \rho^{\mathsf{A}}) \circ \rho^{\mathsf{A}} = (\Delta_{\mathbb{T}} \otimes \mathbf{id}) \circ \rho^{\mathsf{A}}$.
- For $\mathsf{A}, \mathsf{B} \in \mathrm{Ob}(\mathfrak{C}^*_{\mathbb{T}})$ the morphisms between A and B in $\mathfrak{C}^*_{\mathbb{T}}$ are those $\Phi \in \mathrm{Mor}(\mathsf{A}, \mathsf{B})$ such that
$$(\mathbf{id} \otimes \Phi) \circ \rho^{\mathsf{A}} = \rho^{\mathsf{B}} \circ \Phi$$
and the set of those morphisms will be denoted $\mathrm{Mor}_{\mathbb{T}}(\mathsf{A}, \mathsf{B})$.
- The set of **homogeneous** elements, i.e. $a \in \mathsf{A}$ such that $\rho^{\mathsf{A}}(a) = \boldsymbol{z}^n \otimes a$ for some n, spans a dense subspace of A.

- Choose $\zeta \in \mathbb{T}$.
- There exists a monoidal structure $\boxtimes_\zeta$ on $\mathfrak{C}^*_{\mathbb{T}}$ such that
  - for $\mathsf{A}, \mathsf{B} \in \mathrm{Ob}(\mathfrak{C}^*_{\mathbb{T}})$ the C*-algebra $\mathsf{A} \boxtimes_\zeta \mathsf{B}$ comes equipped with embeddings

$$\jmath_1 : \mathsf{A} \longrightarrow \mathsf{A} \boxtimes_\zeta \mathsf{B},$$
$$\jmath_2 : \mathsf{B} \longrightarrow \mathsf{A} \boxtimes_\zeta \mathsf{B}$$

    such that for homogeneous elements

$$\jmath_2(b)\jmath_1(a) = \overline{\zeta}^{\deg(a)\deg(b)} \jmath_1(a)\jmath_2(b),$$

  - the action of $\mathbb{T}$ on $\mathsf{A} \boxtimes_\zeta \mathsf{B}$ is determined uniquely by the condition that $\jmath_1$ and $\jmath_2$ are equivariant.

- For $\Phi\in\operatorname{Mor}_{\mathbb{T}}(\mathsf{A},\mathsf{A}')$ and $\Psi\in\operatorname{Mor}_{\mathbb{T}}(\mathsf{B},\mathsf{B}')$ there exists a unique $\Phi\boxtimes_{\zeta}\Psi\in\operatorname{Mor}_{\mathbb{T}}(\mathsf{A}\boxtimes_{\zeta}\mathsf{B},\mathsf{A}'\boxtimes_{\zeta}\mathsf{B}')$ such that

$$(\Phi\boxtimes_{\zeta}\Psi)\big(\jmath_1(a)\jmath_2(b)\big)=\jmath_1\big(\Phi(a)\big)\jmath_2\big(\Psi(b)\big)$$

  for all $a\in\mathsf{A}$, $b\in\mathsf{B}$.
- If ρ^{B} or ρ^{A} is trivial then

$$\mathsf{A}\boxtimes_{\zeta}\mathsf{B}\cong\mathsf{A}\otimes\mathsf{B}$$

  with $\jmath_1$ and $\jmath_2$ becoming

$$a\longmapsto a\otimes\mathbb{1}\quad\text{and}\quad a\longmapsto\mathbb{1}\otimes b$$

  respectively.

- Fix $q \in \mathbb{C}$ such that $0 < |q| < 1$.
- Define $\mathrm{C}(\mathrm{SU}_q(2))$ to be the universal C*-algebra generated by α and γ such that

$$\begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix}$$

  is unitary.
- The C*-algebra $\mathrm{C}(\mathrm{SU}_q(2))$ is equipped with and action of $\mathbb{T}$ described by

$$\rho^{\mathrm{C}(\mathrm{SU}_q(2))} \in \mathrm{Mor}\big(\mathrm{C}(\mathrm{SU}_q(2)), \mathrm{C}(\mathbb{T}) \otimes \mathrm{C}(\mathrm{SU}_q(2))\big)$$

  determined by

$$\rho^{\mathrm{C}(\mathrm{SU}_q(2))}(\alpha) = \mathbb{1} \otimes \alpha, \quad \rho^{\mathrm{C}(\mathrm{SU}_q(2))}(\gamma) = \boldsymbol{z} \otimes \gamma.$$

THEOREM (KASPRZAK-MEYER-ROY-WORONOWICZ)

Let $\zeta = q/\overline{q}$. There exists a unique

$$\Delta \in \mathrm{Mor}_{\mathbb{T}}\big(\mathrm{C}(\mathrm{SU}_q(2)), \mathrm{C}(\mathrm{SU}_q(2)) \boxtimes_\zeta \mathrm{C}(\mathrm{SU}_q(2))\big)$$

such that

$$\Delta(\alpha) = \jmath_1(\alpha)\jmath_2(\alpha) - q\jmath_1(\gamma^*)\jmath_2(\gamma), \quad \Delta(\gamma) = \jmath_1(\gamma)\jmath_2(\alpha) + \jmath_1(\alpha^*)\jmath_2(\gamma).$$

Moreover

1. *Δ is coassociative:* $(\Delta \boxtimes_\zeta \mathrm{id})\circ\Delta = (\mathrm{id} \boxtimes_\zeta \Delta)\circ\Delta$,
2. *the subspaces*

$$\Delta\big(\mathrm{C}(\mathrm{SU}_q(2))\big)\jmath_2\big(\mathrm{C}(\mathrm{SU}_q(2))\big), \quad \jmath_1\big(\mathrm{C}(\mathrm{SU}_q(2))\big)\Delta\big(\mathrm{C}(\mathrm{SU}_q(2))\big)$$

   are dense in $\mathrm{C}(\mathrm{SU}_q(2)) \boxtimes_\zeta \mathrm{C}(\mathrm{SU}_q(2))$.

- In what follows we will sometimes write $\mathbb{G}$ instead of $\mathrm{SU}_q(2)$.

- Consider the C*-algebra $\mathrm{C}(\mathbb{T})$ with **trivial** action of $\mathbb{T}$.
- Then the map $\pi : \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{T})$ given by

$$\pi(\alpha) = \boldsymbol{z}, \quad \pi(\gamma) = 0$$

is equivariant and we have $(\pi \boxtimes_\zeta \pi)\circ\Delta = \Delta_{\mathbb{T}}\circ\pi$.
- Define $\mathrm{C}(\mathbb{S}^2_q) = \left\{a \in \mathrm{C}(\mathbb{G}) \,\middle|\, (\mathrm{id} \boxtimes_\zeta \pi)\Delta(a) = a \otimes \mathbb{1}\right\}$.

PROPOSITION

1. $(\pi \boxtimes_\zeta \pi)\circ\Delta = \Delta_{\mathbb{T}}\circ\pi$,
2. $\mathrm{C}(\mathbb{S}^2_q)$ is the C*-subalgebra of generated by $\alpha\gamma^*$ and $\gamma^*\gamma$,
3. $\mathrm{C}(\mathbb{S}^2_q)$ is the universal unital C*-algebra generated by elements A and B with relations

$$B^*B = A - A^2, \quad BA = |q|^2 AB, \quad BB^* = |q|^2 A - |q|^4 A^2, \quad A^* = A,$$

4. $\mathrm{C}(\mathbb{S}^2_q)$ is isomorphic to the minimal unitization of the compacts.

- Put $\Gamma = \Delta\big|_{\mathrm{C}(\mathbb{S}^2_q)}$. Then
  - $\Gamma \in \mathrm{Mor}_{\mathbb{T}}(\mathrm{C}(\mathbb{S}^2_q), \mathrm{C}(\mathbb{G}) \boxtimes_\zeta \mathrm{C}(\mathbb{S}^2_q))$,
  - $(\mathrm{id} \boxtimes_\zeta \Gamma)\circ\Gamma = (\Delta \boxtimes_\zeta \mathrm{id})\circ\Gamma$,
  - $\jmath_1\big(\mathrm{C}(\mathbb{G})\big)\Gamma\big(\mathrm{C}(\mathbb{S}^2_q)\big)$ is dense in $\mathrm{C}(\mathbb{G}) \boxtimes_\zeta \mathrm{C}(\mathbb{S}^2_q)$,
  - for $x \in \mathrm{C}(\mathbb{S}^2_q)$ we have $\Gamma(x) = \jmath_2(x)$ if and only if $x \in \mathbb{C}\mathbb{1}$.

- Now define

$$\boldsymbol{V} = \begin{bmatrix} \alpha^2 & -s^2\gamma^*\alpha & -q{\gamma^*}^2 \\ \zeta\alpha\gamma & \mathbb{1} - s^2\gamma^*\gamma & \gamma^*\alpha^* \\ -q\zeta\gamma^2 & -s^2\alpha^*\gamma & {\alpha^*}^2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_{-1,-1} & \boldsymbol{v}_{-1,0} & \boldsymbol{v}_{-1,1} \\ \boldsymbol{v}_{0,-1} & \boldsymbol{v}_{0,0} & \boldsymbol{v}_{0,1} \\ \boldsymbol{v}_{1,-1} & \boldsymbol{v}_{1,0} & \boldsymbol{v}_{1,1} \end{bmatrix}.$$

(with $s = \sqrt{1 + |q|^2}$). Putting $e_i = \boldsymbol{v}_{i,0}$ ($i = -1, 0, 1$) we have
  - $\mathrm{C}(\mathbb{S}^2_q) = \mathrm{C}^*(e_{-1}, e_0, e_1)$,
  - $\Gamma(e_i) = \sum\limits_{j=-1}^{1} \jmath_1(e_i)\jmath_2(\boldsymbol{v}_{i,j})$.

- $\boldsymbol{V}$ is unitary.
- Elements of the matrix $\boldsymbol{V}$ satisfy $\Delta(\boldsymbol{v}_{i,j}) = \sum_{k=-1}^{1} \jmath_1(\boldsymbol{v}_{i,k})\jmath_2(\boldsymbol{v}_{k,j})$.
- If $A \in M_3(\mathbb{C})$ commutes with $\boldsymbol{V}$ then $A \in \mathbb{C}\mathbb{1}_3$.

Moreover

- $e_i{}^* = e_{-i}$ for $i = -1, 0, 1$,
- $\deg(e_i) = i$ for $i = -1, 0, 1$,
- $\mathrm{span}\{e_{-1}, e_0, e_1\}$ is the unique subspace of $\mathrm{C}(\mathbb{S}_q^2)$ equipped with a basis which transforms according to $\boldsymbol{V}$.

Let $\mathbb{X}$ be a compact quantum space such that $\mathrm{C}(\mathbb{X}) \in \mathrm{Ob}(\mathfrak{C}^*_{\mathbb{T}})$. Assume that there exists $\Gamma \in \mathrm{Mor}_{\mathbb{T}}(\mathrm{C}(\mathbb{X}), \mathrm{C}(\mathbb{G}) \boxtimes_\zeta \mathrm{C}(\mathbb{X}))$ such that

- if $a \in \mathrm{C}(\mathbb{X})$ satisfies $\Gamma(a) = \jmath_2(a)$ then $a \in \mathbb{C}\mathbb{1}$,
- $\mathrm{C}(\mathbb{X})$ is generated by a three dimensional subspace $\mathsf{W} \subset \mathrm{C}(\mathbb{X})$ equipped with a basis $\{e_{-1}, e_0, e_1\}$ such that

$$\Gamma(e_i) = \sum_{j=-1}^{1} \jmath_1(e_i)\jmath_2(\boldsymbol{v}_{i,j}),$$

- W is the only subspace of $\mathrm{C}(\mathbb{X})$ equipped with a basis which transforms according to $\boldsymbol{V}$,
- $\deg(e_i) = i$.

(Note that $\mathbb{X} = \mathbb{S}^2_q$ satisfies these conditions.)

Then (after re-scaling $\{e_{-1}, e_0, e_1\}$ by a constant) we have

1. $e_i{}^* = e_{-i}$,
2. there exists $\varrho \in \mathbb{R}$ such that

$$e_{-1}e_1 + s^2 e_0^2 + |q|^2 e_1 e_{-1} = \varrho \mathbb{1},$$

3. there exists $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} s^2(e_{-1}e_0 - |q|^2 e_0 e_{-1}) &= \lambda e_{-1}, \\ |q|^2(e_1 e_{-1} - e_{-1}e_1) + \bigl(1 - |q|^4\bigr)e_0^2 &= \lambda e_0, \\ s^2(e_0 e_1 - |q|^2 e_1 e_0) &= \lambda e_1. \end{aligned}$$

THEOREM

Let $\mathbb{X}_{q,\varrho,\lambda}$ be the universal C*-*algebra generated by e_{-1}, e_0, e_1 with relations ${e_i}^* = e_{-i}$ and*

$$
\begin{aligned}
e_{-1}e_1 + s^2 e_0^2 + |q|^2 e_1 e_{-1} &= \varrho\mathbb{1},\\
s^2(e_{-1}e_0 - |q|^2 e_0 e_{-1}) &= \lambda e_{-1},\\
|q|^2(e_1 e_{-1} - e_{-1}e_1) + \big(1 - |q|^4\big)e_0^2 &= \lambda e_0,\\
s^2(e_0 e_1 - |q|^2 e_1 e_0) &= \lambda e_1.
\end{aligned}
$$

Then

1. *there is an action of $\mathbb{T}$ on $\mathrm{C}(\mathbb{X}_{q,\varrho,\lambda})$ such that $\deg(e_i) = i$,*
2. *There exists $\Gamma_{q,\varrho,\lambda} : \mathrm{C}(\mathbb{X}_{q,\varrho,\lambda}) \to \mathrm{C}(\mathbb{G}) \boxtimes_\zeta \mathrm{C}(\mathbb{X}_{q,\varrho,\lambda})$ such that*

$$
\Gamma_{q,\varrho,\lambda}(e_i) = \sum_{j=-1}^{1} \jmath_1(\boldsymbol{v}_{i,j})\jmath_2(e_j),
$$

3. $(\mathrm{id} \boxtimes_\zeta \Gamma_{q,\varrho,\lambda})\circ\Gamma_{q,\varrho,\lambda} = (\Delta \boxtimes_\zeta \mathrm{id})\circ\Gamma_{q,\varrho,\lambda}$ *and* $\jmath_1\big(\mathrm{C}(\mathbb{G})\big)\Gamma_{q,\varrho,\lambda}\big(\mathrm{C}(\mathbb{X}_{q,\varrho,\lambda})\big)$ *is dense in* $\mathrm{C}(\mathbb{G}) \boxtimes_\zeta \mathrm{C}(\mathbb{X}_{q,\varrho,\lambda})$.

Moreover

We have

$$\mathrm{C}(\mathbb{X}_{q,\varrho,\lambda}) \cong \mathrm{C}(S^2_{|q|c}),$$

where $S^2_{|q|c}$ is the Podleś sphere for $\mathrm{SU}_{|q|}(2)$ and

$$c = |q|^2 \frac{s^2\varrho(1-|q|^2)^2-\lambda^2}{(1+|q|^2)^2\lambda^2}$$

(or $c = \infty$ if $\lambda = 0$).

Conclusion

- Any quantum space with action of $\mathrm{SU}_q(2)$ with properties similar to that of the action on $\mathbb{S}^2_q$ is equivariantly isomorphic to one of the spaces $\mathbb{X}_{q,\lambda,\varrho}$.
- An equivariant isomorphism of $\mathrm{C}(\mathbb{X}_{q,\lambda,\varrho})$ onto $\mathrm{C}(\mathbb{X}_{q,\lambda',\rho'})$ must map the distinguished basis $\{e_{-1},e_0,e_1\}$ of $\mathsf{W}\subset\mathrm{C}(\mathbb{X}_{q,\lambda,\varrho})$ to a multiple of the distinguished basis of $\mathsf{W}'\subset\mathrm{C}(\mathbb{X}_{q,\lambda',\varrho'})$.
- The collection of compact quantum spaces $\{\mathbb{X}_{q,\lambda,\varrho}\}$ coincides with $\{S^2_{|q|c}\}_{c\in\mathbb{R}\cup\{\infty\}}$.