Mathematics and physics of sound and music

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Pythagorean theorem

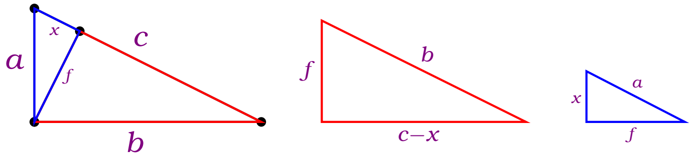
Theorem (First version)

In a right triangle with legs a and b and hypotenuse c we have

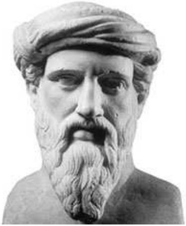
$$a^2 + b^2 = c^2.$$

Theorem (second version)

The altitude of a right triangle divides it into two triangles similar to the original.



Pythagoras of Samos

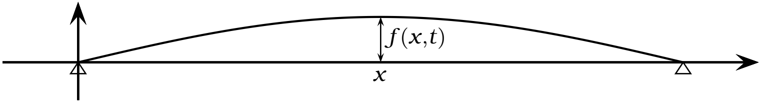


- Born around 570 BCE, died around 495 BCE
- Philosopher, mathematician, musician, ...
- Founded and lead a secret organization — the “Pythagorean brotherhood”
- Pythagoras and his followers believed that the world should be described by mathematics and, in particular, its highest form — geometry.
- He is quoted to have said that “all things are numbers”, with the word “number” meaning “rational number” or “ratio”.

Vibrating string

- Homogeneous string of length ℓ m, mass density ρ $\frac{\text{kg}}{\text{m}}$ and tension T N

- Deviation from equilibrium at point x at time t:



- From Newton's law ($m\vec{a} = \vec{F}$) we derive the equation which is satisfied by f

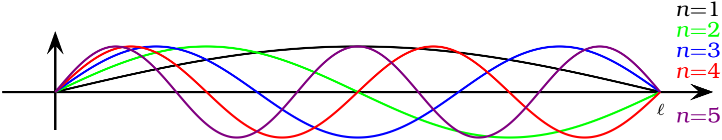
$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}, \qquad \left(\begin{smallmatrix}\text{wave}\\ \text{equation}\end{smallmatrix}\right)$$

where $c = \sqrt{T/\rho}$ (velocity of wave propagation).

Vibrating string

- Some solutions of the wave equation:

$$f(x,t) = \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi ct}{\ell} + \varphi\right).$$



$(t = \varphi = 0)$.

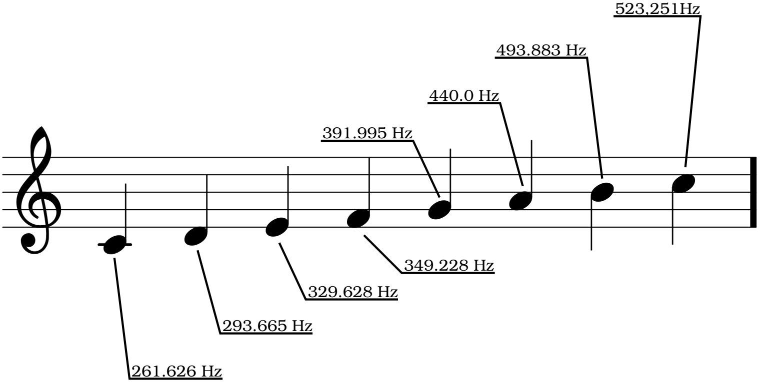
- Any other solution is a combination (sum) of such solutions.
- The **frequency** ν is

$$\nu = \frac{n}{2\ell}\sqrt{\frac{T}{\rho}}.$$

- Usually we hear the sound of $n = 1$, higher frequencies are softer (we call them **overtones**).

Musical sounds

- Vibrating strings produce sounds used when making music.
- Usually we limit ourselves to particular frequencies, e.g.



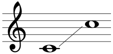
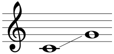
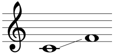
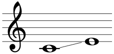
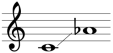
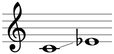
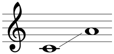
Intervals

- The "distances" between sounds we hear are called **intervals**.
- Our brains perceive the interval through the ratio of frequencies: the interval from a sound of frequency 261 Hz (c^1) to the sound of frequency 329 Hz (e^1) is the same as from f^1 (349 Hz) to a^1 (440 Hz), because

$$\frac{329}{261} \approx 1.260 \approx \frac{440}{349}.$$

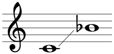
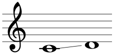
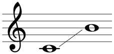
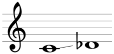
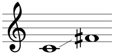
- Thus intervals are nothing else than Pythagorean ratios.
- Intervals for which the ratios are rational numbers with small denominator sound to our ears "in tune" or "harmonious" and we call them **consonances**.
- Intervals without that property are called **dissonances**.

Consonances

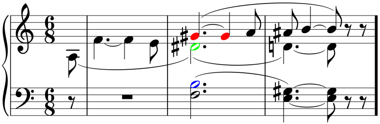
Interval	**Ratio***	**Example**
Octave	2 : 1	
Fifth	3 : 2	
Fourth	4 : 3	
Major third	5 : 4	
Minor sixth	8 : 5	
Minor third	6 : 5	
Major sixth	5 : 3	

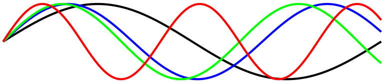
*In practice the ratios are slightly different.

Dissonances

Interval	Ratio	Example
Minor seventh	$\approx 16:9$	
Major second	$\approx 10:9$	
Major seventh	$\approx 15:8$	
Minor second	$\approx 16:15$	
Tritone	$\approx 45:32$	

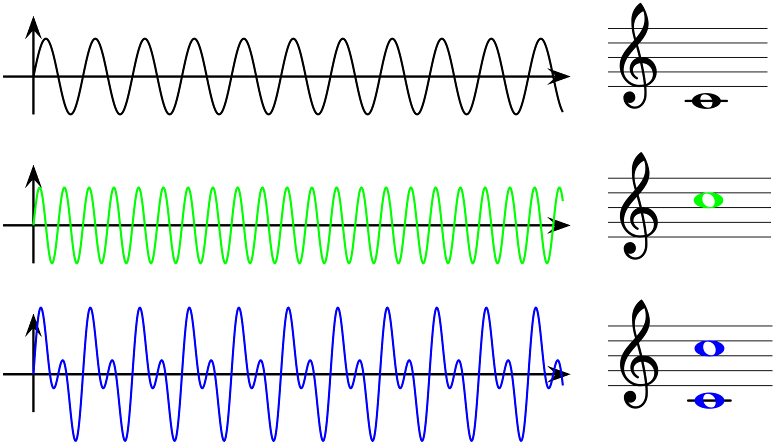
- Instead of playing separate notes, we can play them together:



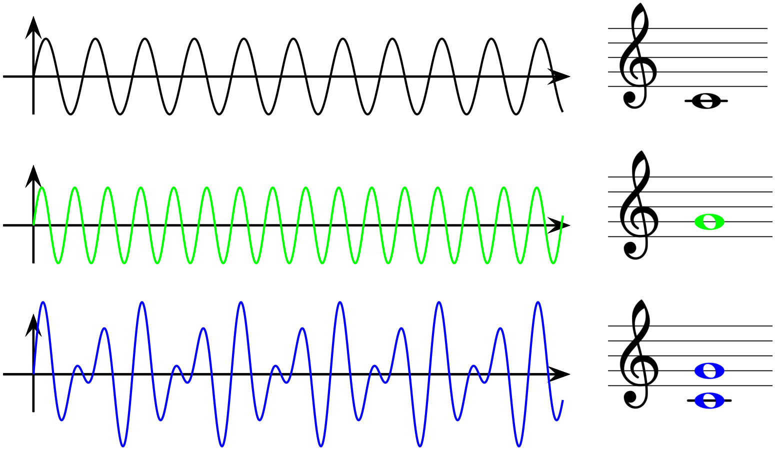


- We then hear different sound waves overlapping.
- This means that the solutions of the wave equations **add up**.
- This phenomenon is called **interference** of waves.
- Let us see what that looks like for different intervals.

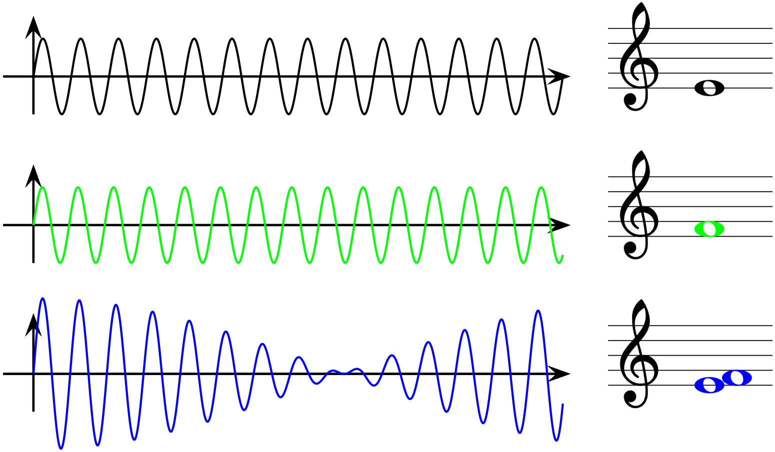
Octave



Fifth

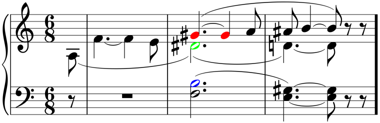


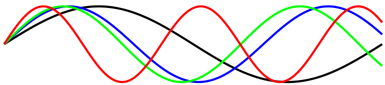
Minor second



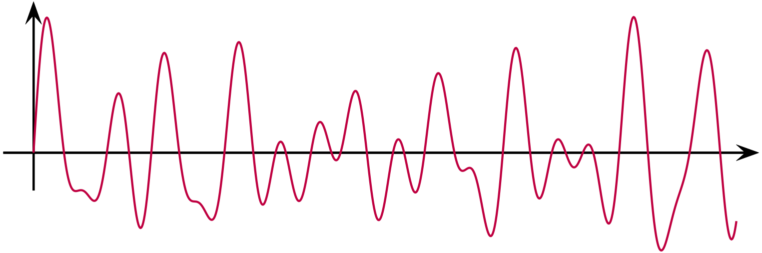
The Tristan chord

- Recall the opening chord form “Tristan und Isolde”:





- Here is the wave pattern:

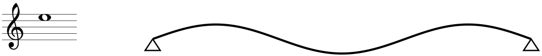


Harmonic series

- Start with a string vibrating with frequency 220 Hz (the note a):

- we also hear a softer note a^1 (440 Hz):

- and e^2 (660 Hz):



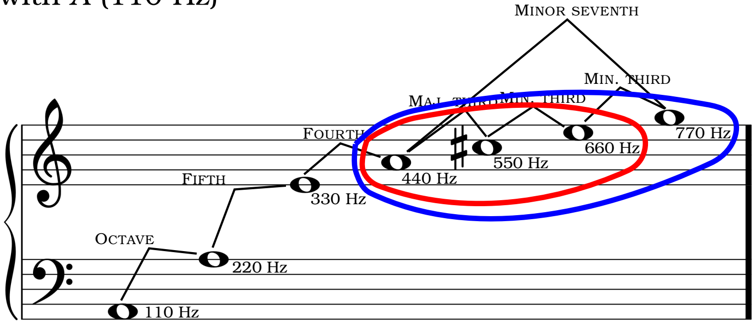
- as well as a^2 (880 Hz):

- and $c\sharp^3$ (1100 Hz):

- etc.

Harmonic series

- Let us write all the overtones on one staff starting with A (110 Hz)



- The notes marked in red form the **A major chord**.
- The ones marked in blue form the **seventh chord**.
- The common major-minor system is therefore a consequence of the basic physics of sound.