Analysis on the quantum plane - a step towards Schwartz space for the $E_q(2)$ group

(A report on research in progress)

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Motivation: Gelfand's method of constructing representations of the Lorentz group was succesfully applied to the quantum Lorentz group in [PuSLW] (see also [PoSLW]). These are the representations induced from the one-dimensional representations of the parabolic subgroup.

In an attempt to apply Gelfand's construction to the quantum Lorentz group with Gauss decomposition (see [ZaSLW]) we need to find apropriate carrier spaces which classically should correspond to some spaces of smooth functions on $SL(2,\mathbb{C})$. We want to start with the definition of the Schwartz space $\mathcal{S}\left(E_q(2)\right)$ for the group $E_q(2)$ with the property that the comultiplication on $E_q(2)$ maps $\mathcal{S}\left(E_q(2)\right)$ into $\mathcal{S}\left(E_q(2)\right)\widehat{\otimes}\mathcal{S}\left(E_q(2)\right)$.

(Quite surprisingly) by the quantum plane we mean the set

$$\overline{\mathbb{C}}^q = \{z \in \mathbb{C} \colon z = 0 \text{ or } |z| \in q^{\mathbb{Z}}\}$$

(where $0 < q < 1$ is a parameter) or rather the C^*-algebra

$$A = C_\infty\big(\overline{\mathbb{C}}^q\big).$$

A has a braided group structure — see the paper [SLW6]

It makes sense to call it a (quantum) deformation of $\mathbb{R}^2$.

Some notation:

$$A^\eta = C\left(\overline{\mathbb{C}}^q\right)$$

A^η is the set of elements affiliated with A — see the paper [SLW2].

A^η is a topological $*$-algebra. The following automorphisms of A will play an essential role:

$$\begin{array}{rcl} \sigma_t &:& (\sigma_t(f))(z) = f(q^{it}z), \\ \tau &:& (\tau(f))(z) = f(qz). \end{array}$$

The strongly continuous group σ_t and the automorphism τ obviously extend to A^η.

Let $\mathcal{A}^\eta$ be the set of entire analytic elements for the group σ_t and let σ_i be its analytic generator (see [PMS] and references therein). $\mathcal{A}^\eta$ is a $*$-algebra. In particular any function $f \in \mathcal{A}^\eta$ has the property that its restriction to any circle of $\overline{\mathbb{C}}^q$ has a holomorphic extension to $\mathbb{C} \setminus \{0\}$.

<u>Strange notation!</u>

Explained in this table:

$$\begin{array}{rcl} f & \longleftrightarrow & f(z) \\ \tau(f) & \longleftrightarrow & f(qz) \\ \sigma_t(f) & \longleftrightarrow & f(q^{it} z) \\ \sigma_i(f) & \longleftrightarrow & f(q^{-1} \cdot z) \\ \sigma_{-i}(f) & \longleftrightarrow & f(q \cdot z) \\ (\sigma_i \circ \tau)(f) & \longleftrightarrow & f(q^{-1} \cdot qz) \end{array}$$

We shall construct a first order differential $*$-calculus (Γ, d) over $\mathcal{A}^\eta$ (see [SLW1]).

Let Γ be the $\mathcal{A}^\eta$-bimodule generated by two distinguished elements

$$\omega \quad \text{and} \quad \overline{\omega}$$

with relations

$$\left.\begin{aligned} f(z)\omega &= \omega f(q \cdot qz), \\ f(z)\overline{\omega} &= \overline{\omega} f(q \cdot q^{-1}z), \end{aligned}\right\} \qquad (1)$$

for all functions $f \in \mathcal{A}^\eta$.

Setting $\omega^* = \overline{\omega}$ we obtain a $\mathcal{A}^\eta$-$*$-bimodule structure on Γ.

Now we define $d \colon \mathcal{A}^\eta \to \Gamma$:

$$df = [\omega - \overline{\omega}, f] = (\omega - \overline{\omega})f - f(\omega - \overline{\omega}).$$

We apply d to functions $z \mapsto z$ and $z \mapsto \overline{z}$:

$$\begin{aligned} dz &= [\omega - \overline{\omega}, z] = (q^{-2} - 1) z\omega, \\ d\overline{z} &= [\omega - \overline{\omega}, \overline{z}] = (q^{-2} - 1)\overline{\omega}\,\overline{z}. \end{aligned}$$

And now we can rewrite the differential d in terms of dz and $d\overline{z}$:

$$\begin{aligned} df &= \frac{\partial_{\mathrm{L}} f}{\partial z} dz + d\overline{z} \frac{\partial_{\mathrm{R}} f}{\partial \overline{z}} \\ &= dz \frac{\partial_{\mathrm{R}} f}{\partial z} + \frac{\partial_{\mathrm{L}} f}{\partial \overline{z}} d\overline{z}, \end{aligned}$$

where the differential operators are found using the relations (1).

For $f \in \mathcal{A}^\eta$ we have:

$$\begin{aligned}
\frac{\partial_{\rm L} f}{\partial z} &= \frac{q^2}{1-q^2}\,\frac{f(q^{-1}\cdot q^{-1}z)-f(z)}{z}\\
\frac{\partial_{\rm R} f}{\partial z} &= \frac{1}{1-q^2}\,\frac{f(z)-f(q\cdot qz)}{z}\\
\frac{\partial_{\rm L} f}{\partial \overline{z}} &= \frac{1}{1-q^2}\,\frac{f(z)-f(q^{-1}\cdot qz)}{z}\\
\frac{\partial_{\rm R} f}{\partial \overline{z}} &= \frac{q^2}{1-q^2}\,\frac{f(q\cdot q^{-1}z)-f(z)}{\overline{z}}
\end{aligned}$$

It is worth noting that for $f \in \mathcal{A}^\eta$

$$\left(\begin{array}{c} f \text{ is a restriction to } \overline{\mathbb{C}}^q \\ \text{of an entire function}\end{array}\right) \Longleftrightarrow \left(\frac{\partial_{\rm L} f}{\partial \overline{z}} = 0\right)$$

(and the same for $\frac{\partial_{\rm R}}{\partial \overline{z}}$).

From the Leibnitz rule for d we get the following formula:

$$\frac{\partial_{\mathrm{R}}(fg)}{\partial\overline{z}} = \frac{\partial_{\mathrm{R}}f}{\partial\overline{z}}g + f(q\cdot q^{-1}z)\frac{\partial_{\mathrm{R}}g}{\partial\overline{z}}$$

for $f, g \in \mathcal{A}^{\eta}$, so if g is an entire function we get

$$\frac{\partial_{\mathrm{R}}(fg)}{\partial\overline{z}} = \frac{\partial_{\mathrm{R}}f}{\partial\overline{z}}g.$$

Using this equation we can easily compute the derivative of the special function F_q (see [SLW4]):

$$\frac{\partial_{\mathrm{R}}F_q(\lambda z)}{\partial\overline{z}} = \frac{\overline{\lambda}}{1-q^2}F_q(\lambda z), \quad \lambda, z \in \overline{\mathbb{C}}^q.$$

Integration over $\overline{\mathbb{C}}^q$ — definition:

$$\int\limits_{\overline{\mathbb{C}}^q} f(z)d\mu(z) = \sum_{k=-\infty}^{+\infty} q^{2k}\frac{1}{2\pi}\int\limits_0^{2\pi} f(e^{i\theta}q^k)d\theta = \sum_{k=-\infty}^{+\infty} q^{2k}\int\limits_{q^kS^1} f(z)\frac{dz}{2\pi iz}.$$

We have an analogue of Stoke's theorem, namely if $f \in \mathcal{A}^\eta$ has compact support then

$$\int\limits_{\overline{\mathbb{C}}^q} \frac{\partial_{\mathrm{R}} f}{\partial \overline{z}} d\mu = 0.$$

We also have the formulae:

$$\int\limits_{\overline{\mathbb{C}}^q} f(qz)d\mu(z) = q^{-2}\int\limits_{\overline{\mathbb{C}}^q} f(z)d\mu(z),$$
$$\int\limits_{\overline{\mathbb{C}}^q} f(q\cdot z)d\mu(z) = \int\limits_{\overline{\mathbb{C}}^q} f(z)d\mu(z).$$

With these formulae we can prove that

$$\frac{\partial_{\mathrm{L}}}{\partial z}^* = -q^2\frac{\partial_{\mathrm{L}}}{\partial \overline{z}},$$
$$\frac{\partial_{\mathrm{R}}}{\partial z}^* = -q^{-2}\frac{\partial_{\mathrm{R}}}{\partial \overline{z}}$$

on $L^2\big(\overline{\mathbb{C}}^q\big) = L^2\big(\overline{\mathbb{C}}^q,\mu\big)$.

Furthermore it can be proved that these are closed normal operators on $L^2\big(\overline{\mathbb{C}}^q\big)$ with a common core consisting of finite linear combinations of the functions $g_{kl}(z) = \chi(|z| = q^k)\,(\mathrm{Phase}\,z)^l$.

The Fourier transform:

$$\mathcal{F}f(z) = \int\limits_{\overline{\mathbb{C}}^q} F_q(\lambda z) f(z) d\mu(z).$$

For $k, m \in \mathbb{Z}$ and $l \in \mathbb{Z}_+$ we introduce the seminorms:

$$\|f\|_{k,m,l} = \left\| z^k \left(\frac{\partial_{\mathrm{R}}}{\partial \overline{z}} \right)^l f(q^m \cdot z) \right\|_{L^2}$$

And define $\mathcal{S}\big(\overline{\mathbb{C}}^q\big)$ as the space of those functions from $\mathcal{A}^\eta$ all of whose $\|\cdot\|_{k,l,m}$ seminorms are finite:

$$\mathcal{S}\big(\overline{\mathbb{C}}^q\big) = \{f \in \mathcal{A}^\eta \colon \|f\|_{k,l,m} < \infty \, \forall k, l, m\}$$

Thanks to the formulae:

$$\frac{\partial_{\mathrm{R}}}{\partial \overline{z}} (\mathcal{F} f)(z) = \frac{1}{1-q^2} \left(\mathcal{F}\left(\overline{\lambda} f\right)\right)(z),$$

$$\left(\mathcal{F}\left(\frac{\partial_{\mathrm{L}} f}{\partial \overline{\lambda}}\right)\right)(z) = -\frac{q^{-2}}{1-q^2} \overline{z} (\mathcal{F} f)(z)$$

and the formula

$$\frac{\partial_{\mathrm{R}}}{\partial \overline{z}} f(q^{-1} \cdot qz) = q^2 \frac{\partial_{\mathrm{L}} f}{\partial \overline{z}} \qquad (2)$$

We see that $\mathcal{F}$ is a topological isomorphism of $\mathcal{S}\left(\overline{\mathbb{C}}^q\right)$ onto itself.

The normality of $\frac{\partial_{\mathrm{R}}}{\partial \overline{z}}$ and formula (2) allows us to forget about all the other differential operatrs we have defined.

Woronowicz proved that the space $\mathcal{S}\left(\overline{\mathbb{C}}^q\right)$ is nuclear.

Back to quantum groups — the $E_q(2)$ group is a C^*-algebra generated by two elements affiliated with it (see [SLW7]) n and v with relations

$$vnv^* = qn.$$

It is in fact the crossed product $C_\infty\left(\overline{\mathbb{C}}^q\right) \times \mathbb{Z}$ and consists of sums of the form

$$\sum_{k \in \mathbb{Z}} v^k f_k(n)$$

where the f_k's belong to $C_\infty\left(\overline{\mathbb{C}}^q\right)$.

We are now investigating various conditions on sums of the form

$$\sum_{k\in\mathbb{Z}} v^k f_k(n)$$

with $f_k \in \mathcal{S}\left(\overline{\mathbb{C}}^q\right)$ in order to properly define the space $\mathcal{S}\left(E_q(2)\right)$ i.e. to have

$$\Delta\colon \mathcal{S}\left(E_q(2)\right) \longrightarrow \mathcal{S}\left(E_q(2)\right) \widehat{\otimes} \mathcal{S}\left(E_q(2)\right).$$

where Δ is the comultiplication of $E_q(2)$.

Remark: for a classical group there is no way a condition of that kind be satisfied. Nevertheless there is evidence it could be the right condition for the *quantum* version of $E(2)$ (see [SLW5]).

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