Embeddable quantum homogeneous spaces

C*-algebras and Banach algebras
Instytut Matematyczny
Polskiej Akademii Nauk

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Plan of talk

1. Locally compact quantum groups
2. Quantum $\mathbb{G}$-spaces
3. Closed quantum subgroups and quotients
4. W^*-quantum homogeneous $\mathbb{G}$-spaces
5. Embeddable quantum homogeneous spaces
6. Quotient by the diagonal subgroup

Definition

A **locally compact quantum group** $\mathbb{G}$ consist of a von Neumann algebra M, a normal unital injective map

$$\Delta\colon \mathsf{M} \longrightarrow \mathsf{M}\bar{\otimes}\mathsf{M}$$

such that $(\Delta \otimes \mathrm{id})\circ\Delta = (\mathrm{id} \otimes \Delta)\circ\Delta$, and two n.s.f. weights φ and ψ on M such that

$$\begin{aligned}(\mathrm{id} \otimes \varphi)\Delta(x) &= \varphi(x)\mathbb{1}, \qquad (x \in \mathfrak{M}_\varphi),\\ (\psi \otimes \mathrm{id})\Delta(x) &= \psi(x)\mathbb{1}, \qquad (x \in \mathfrak{M}_\psi).\end{aligned}$$

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- Both $L^\infty(\mathbb{G})$ and $L^\infty(\widehat{\mathbb{G}})$ are naturally represented on the GNS Hilbert space of ψ called $L^2(\mathbb{G})$.

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- If G is not abelian, $L^\infty(\widehat{G})$ is the group von Neumann algebra of G.

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- By analogy with group C^*-algebras, $C_0(\mathbb{G})$ is often called the **reduced** C^*-algebra describing $\mathbb{G}$.

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  are relevant (and desirable).

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- If, moreover, $(\mathrm{C}_0(\mathbb{G}) \otimes \mathbb{1})\alpha(\mathsf{A}) \underset{\text{dense}}{\subset} \mathrm{C}_0(\mathbb{G}) \otimes \mathsf{A}$ then this action is **unital**.

Quantum $\mathbb{G}$-Spaces

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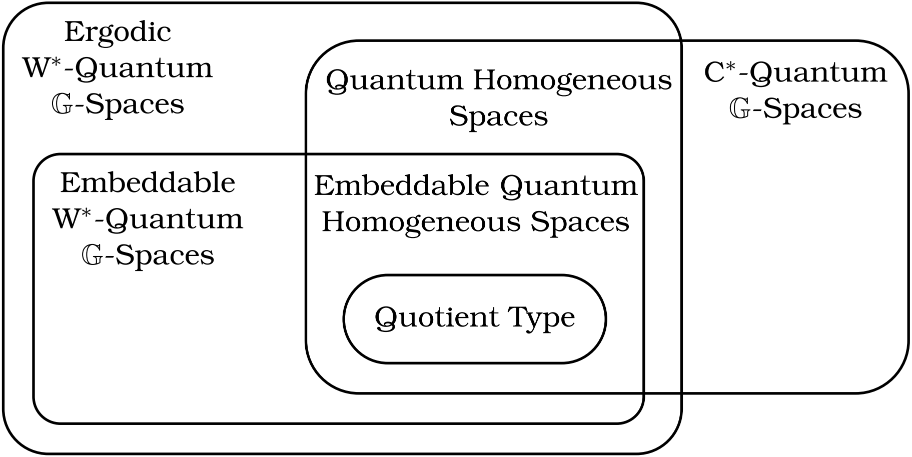
$\mathbb{G}$ — a locally compact quantum group.



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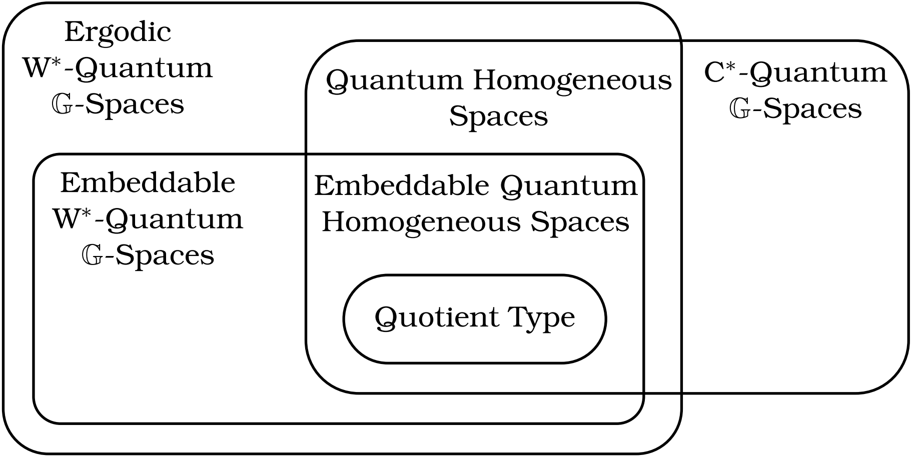

- Many classes of objects, some relations unclear

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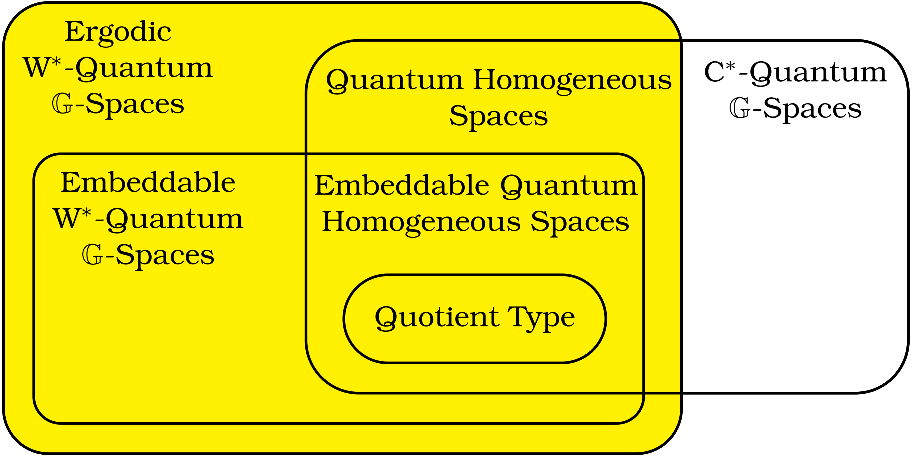

- Von Neumann algebra language, $\alpha(x) = \mathbb{1} \otimes x \;\Rightarrow\; x \in \mathbb{C}\mathbb{1}$

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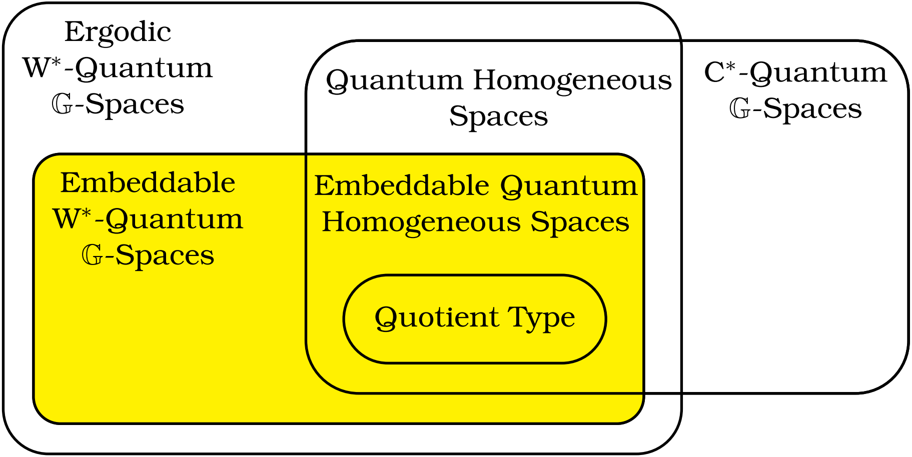

- Left coideals in $L^\infty(\mathbb{G})$, co-duality

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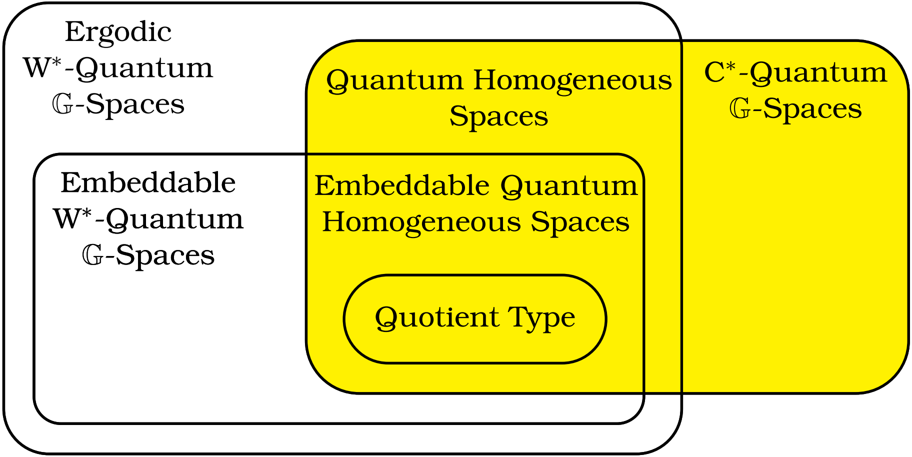

- C*-algebra language, Podleś condition

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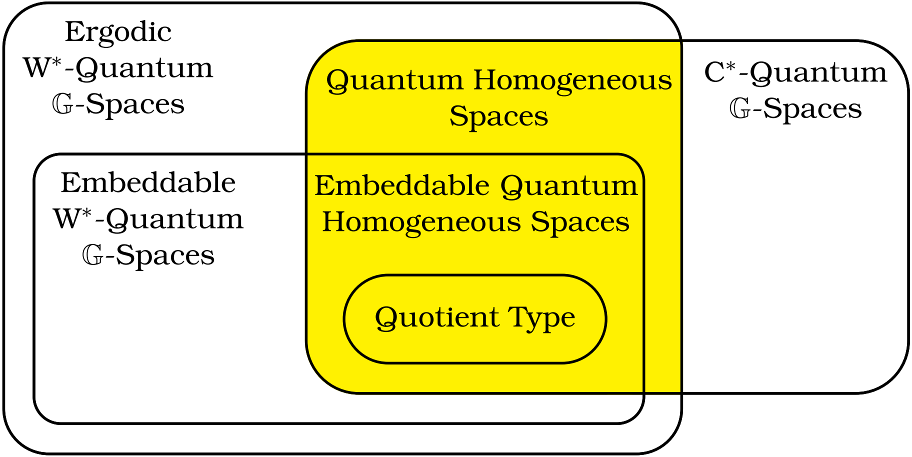

- Compatible C*- and von Neumann description

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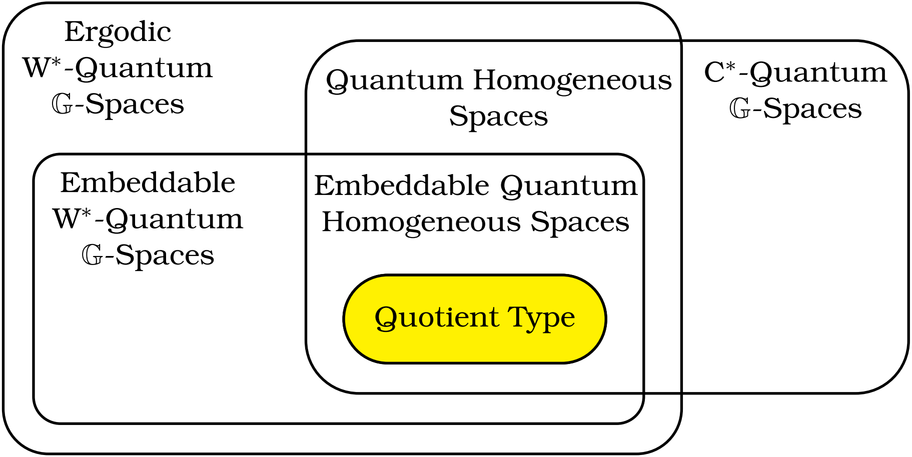

- Defined by S. Vaes, cf. work of P. Podleś

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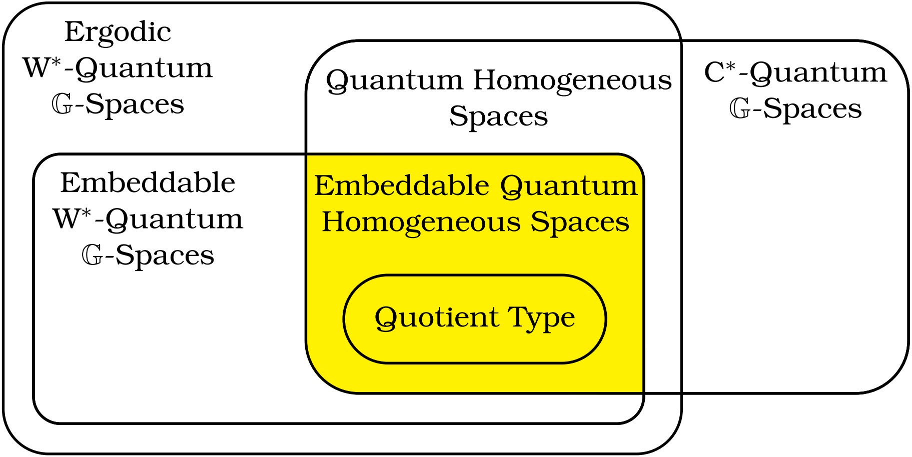

- Natural class we wish to study

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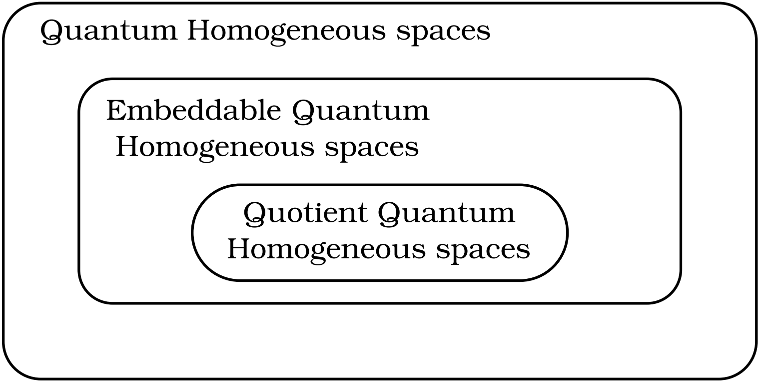

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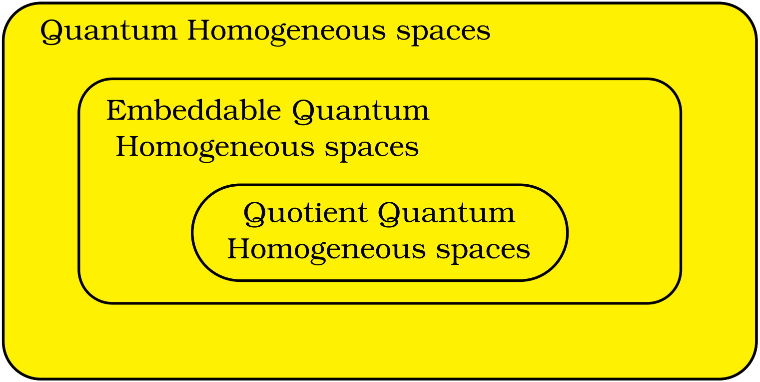

- Ergodic actions (transitivity)

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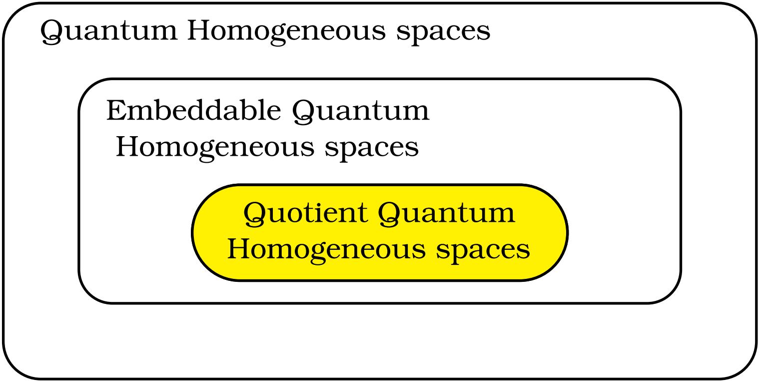

- Q.H.S.'s arising from subgroups (careful)

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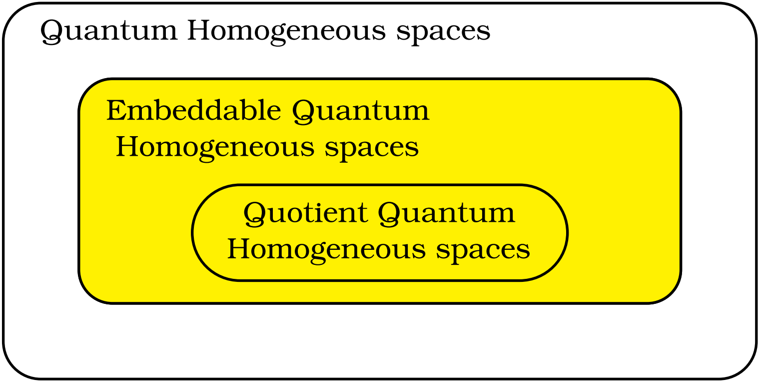

- Ergodic actions realized inside $C(\mathbb{G})$ via Δ

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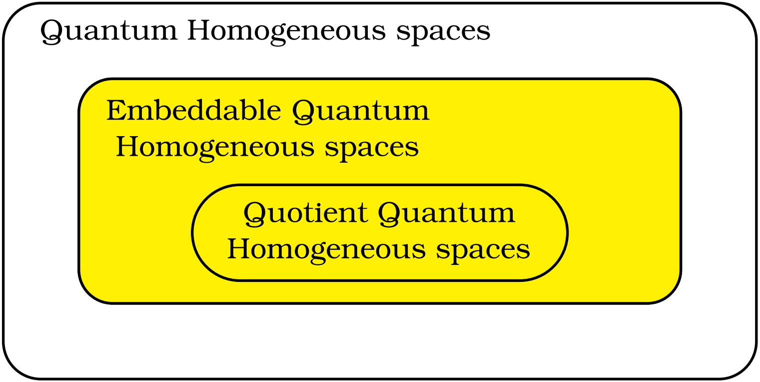

- Classically correspond to classical homogeneous spaces

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- converse unclear, true in many cases.

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- $\mathbb{X}$ is by definition the quotient space $\mathbb{G}/\mathbb{H}$.

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- Define a quantum space $\mathbb{X}$ setting

$$L^\infty(\mathbb{X}) = \bigl\{x \in L^\infty(\mathbb{G}) \bigm| \alpha(x) = x\otimes\mathbb{1}\bigr\}.$$

- $\mathbb{X}$ is by definition the quotient space $\mathbb{G}/\mathbb{H}$.
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Quotient construction

- Let $\mathbb{H}$ be a closed quantum subgroup of $\mathbb{G}$ in the sense of Vaes with $\widehat{\pi}\colon L^\infty(\widehat{\mathbb{H}}) \hookrightarrow L^\infty(\widehat{\mathbb{G}})$
- Define a right action of $\mathbb{H}$ on $\mathbb{G}$ by

$$\alpha(x) = \bigl[(\widehat{\pi}\otimes\mathrm{id})W^{\mathbb{H}}\bigr](\mathbb{1}\otimes x)\bigl[(\widehat{\pi}\otimes\mathrm{id})W^{\mathbb{H}}\bigr]^*,$$

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Embeddable W*-quantum $\mathbb{G}$-spaces

Embeddable W^*-quantum $\mathbb{G}$-spaces

Definition

A quantum space $\mathbb{X}$ is an **embeddable** W^***-quantum** $\mathbb{G}$**-space** if $L^\infty(\mathbb{X}) \subset L^\infty(\mathbb{G})$ and $\Delta_{\mathbb{G}}\big(L^\infty(\mathbb{X})\big) \subset L^\infty(\mathbb{G}) \mathbin{\bar{\otimes}} L^\infty(\mathbb{X})$.

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Theorem (Co-dual)

Define $\widetilde{\mathbb{X}}$ by setting

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Then $\widetilde{\mathbb{X}}$ is an embeddable W**-quantum* $\widehat{\mathbb{G}}$*-space.*

$\widetilde{\mathbb{X}}$ IS A W*-QUANTUM $\widehat{\mathbb{G}}$-SPACE

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PROOF.

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$$\begin{aligned}\Delta_{\widehat{\mathbb{G}}}(y)(\mathbb{1}\otimes x) &= \widehat{W}(y\otimes\mathbb{1})\widehat{W}^*(\mathbb{1}\otimes x)\\ &= \Sigma W^*\Sigma(y\otimes\mathbb{1})\Sigma W\Sigma(\mathbb{1}\otimes x)\end{aligned}$$

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□

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Theorem

$\mathbb{X}$ is of quotient type iff there exists a closed quantum subgroup $\mathbb{H}$ of $\mathbb{G}$ such that $L^\infty(\widetilde{\mathbb{X}})$ is the image of $L^\infty(\widehat{\mathbb{H}})$ in $L^\infty(\widehat{\mathbb{G}})$.

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is **strict**, i.e. strong*–strict continuous on $\|\cdot\|$-bounded subsets.

What?!

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- It is a highly non-trivial theorem of Vaes, that the quotient type W^*-quantum $\mathbb{G}$-spaces are embeddable quantum homogeneous spaces.
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  - moreover, $\mathbb{X}$ is an embeddable quantum homogeneous space for $\mathbb{G} \times \mathbb{G}^{\mathrm{op}}$ with $\mathrm{C}_0(\mathbb{X}) = \Delta_{\mathbb{G}}\big(\mathrm{C}_0(\mathbb{G})\big)$.

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- In particular we find that **quantum** groups do not have diagonal subgroups.