# EMBEDDABLE QUANTUM HOMOGENEOUS SPACES

#### C\*-ALGEBRAS AND BANACH ALGEBRAS INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

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#### July 10, 2013

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# PLAN OF TALK

1 LOCALLY COMPACT QUANTUM GROUPS

- 2 QUANTUM G-SPACES
- 3 CLOSED QUANTUM SUBGROUPS AND QUOTIENTS
- 4 W<sup>\*</sup>-QUANTUM HOMOGENEOUS G-SPACES
- 5 Embeddable quantum homogeneous spaces
- 6 QUOTIENT BY THE DIAGONAL SUBGROUP

#### DEFINITION

A **locally compact quantum group**  $\mathbb{G}$  consist of a von Neumann algebra M, a normal unital injective map

$$\Delta \colon \mathsf{M} \longrightarrow \mathsf{M} \,\bar{\otimes} \,\mathsf{M}$$

such that  $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ , and two n.s.f. weights  $\varphi$  and  $\psi$  on M such that

$$(\mathrm{id}\otimes arphi)\Delta(x) = arphi(x)\mathbb{1}, \qquad (x\in\mathfrak{M}_{arphi}), \ (\psi\otimes\mathrm{id})\Delta(x) = \psi(x)\mathbb{1}, \qquad (x\in\mathfrak{M}_{\psi}).$$

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- Each l.c.q.g.  $\mathbb{G}$  has a **dual**  $\widehat{\mathbb{G}}$  and the dual of  $\widehat{\mathbb{G}}$  is naturally isomorphic to  $\mathbb{G}$ .
- Both L<sup>∞</sup>(G) and L<sup>∞</sup>(G) are naturally represented on the GNS Hilbert space of ψ called L<sup>2</sup>(G).

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### $MOTIVATING \ \mathsf{EXAMPLE}$

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where  $h_{\rm L}$  and  $h_{\rm R}$  are the left and right Haar measures of G.

- If G is abelian then  $\widehat{G}$  is the Pontiagin dual of G.
- If *G* is not abelian,  $L^{\infty}(\widehat{G})$  is the group von Neumann algebra of *G*.

### $C^*$ -Algebraic Approach

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- There is another C\*-algebra  $C_0^u(\mathbb{G})$  with a surjection  $\Lambda: C_0^u(\mathbb{G}) \to C_0(\mathbb{G})$  which encodes representation theory of  $\widehat{\mathbb{G}}; C_0^u(\mathbb{G})$  is called the **universal version** of  $C_0(\mathbb{G})$ .

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- By analogy with group C\*-algebras,  $C_0(\mathbb{G})$  is often called the **reduced** C\*-algebra describing  $\mathbb{G}$ .

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## EXAMPLE REVISITED

- *G* locally compact group.
- For  $f \in C_0(G)$  put

$$(\Delta(f))(s,t) = f(st),$$
  $(s,t \in G).$ 

- Note that  $\Delta(f) \notin C_0(G \times G) \cong C_0(G) \otimes C_0(G)$ .
- Clearly  $\Delta(f) \in C_B(G \times G) \cong M(C_0(G) \otimes C_0(G)).$
- $\Delta(C_0(G))(C_0(G) \otimes C_0(G))$  is dense in  $C_0(G) \otimes C_0(G)$ . ( $\Delta$  is a **morphism**.)
- In fact  $\Delta(C_0(G))(\mathbb{1} \otimes C_0(G))$  and  $\Delta(C_0(G))(C_0(G) \otimes \mathbb{1})$  are dense in  $C_0(G) \otimes C_0(G)$ .
- $C_0^u(G)$  is canonically isomorphic to  $C_0(G)$ .
- $C_0(\widehat{G})$  is the reduced group C\*-algebra of *G*.
- $C_0^u(\widehat{G})$  is the universal group C\*-algebra of *G*.

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QUANTUM HOMOGENEOUS SPACES

•  $\mathbb{G}$  — locally compact quantum group,

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- $\mathbb{G}$  locally compact quantum group,
- $\mathbb{X}$  quantum space, i.e. either
  - ► a C\*-algebra called  $C_0(X)$  is given (topological structure) or
  - ► a v.N. algebra called  $L^{\infty}(\mathbb{X})$  is given (measurable structure)
- $\bullet$  An **action** of  $\mathbb G$  on  $\mathbb X$  is described by either
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• Continuity and Podleś Condition

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- **Continuity** and **Podleś Condition**: in the C\*-context the conditions
  - $(\mathbf{C}_0(\mathbb{G}) \otimes \mathbb{1}) \alpha(\mathbf{C}_0(\mathbb{X})) \subset \mathbf{C}_0(\mathbb{G}) \otimes \mathbf{C}_0(\mathbb{X})$

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PIOTR M. SOŁTAN (WARSAW)

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#### are relevant (and desirable).

PIOTR M. SOŁTAN (WARSAW)

JULY 10, 2013 7 / 20

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QUANTUM HOMOGENEOUS SPACES

JULY 10, 2013 8 / 20

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• Let *G* act on a l.c. space *X*.

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• Given a C\*-algebra A and  $\alpha \in Mor(A, C_0(G) \otimes A)$  s.t. (\*\*), the condition that  $(C_0(G) \otimes 1)\alpha(A) \subset C_0(G) \otimes A$  implies existence of a **continuous** action of *G* on A such that (\*) holds.

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- If, moreover,  $(C_0(\mathbb{G}) \otimes 1)\alpha(A) \underset{\text{dense}}{\subset} C_0(\mathbb{G}) \otimes A$  then this action is **unital**.

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## Quantum $\mathbb{G}$ -Spaces

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## Quantum $\mathbb{G}$ -Spaces

 $\mathbb{G}$  — a locally compact quantum group.

Ergodic W*-Quantum G-Spaces	Quantum Homogeneous Spaces	C*-Quantum G-Spaces
Embeddable W*-Quantum G-Spaces	Embeddable Quantum Homogeneous Spaces Quotient Type	
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 $\mathbb{G}$  — a locally compact quantum group.



Many classes of objects, some relations unclear

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# Quantum $\mathbb{G}$ -Spaces

 $\mathbb{G}$  — a locally compact quantum group.



#### • Von Neumann algebra language, $\alpha(x) = \mathbb{1} \otimes x \Rightarrow x \in \mathbb{C}\mathbb{1}$

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• Left coideals in  $L^{\infty}(\mathbb{G})$ , co-duality

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• C\*-algebra language, Podles condition

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• Compatible C\*- and von Neumann description

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# Quantum $\mathbb{G} ext{-}Spaces$

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• Defined by S. Vaes, cf. work of P. Podleś

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# Quantum $\mathbb{G} ext{-}Spaces$

 $\mathbb{G}$  — a locally compact quantum group.



#### • Natural class we wish to study

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 $\bullet~\mathbb{G}$  — a compact quantum group

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 $\bullet \ \mathbb{G}$  — a compact quantum group (C\_0(\mathbb{G}) is unital).

- $\mathbb{G}$  a compact quantum group ( $C_0(\mathbb{G})$  is unital).
- consider only actions on **compact** quantum spaces.

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• Ergodic actions (transitivity)

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Image: A matrix and a matrix

- $\mathbb{G}$  a compact quantum group (C\_0( $\mathbb{G})$  is unital).
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• Q.H.S.'s arising from subgroups (careful)

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Image: A matrix and a matrix

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• Ergodic actions realized inside  $C(\mathbb{G})$  via  $\Delta$ 

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- ${\ {\rm o} \ } {\mathbb G}$  a compact quantum group (C\_0({\mathbb G}) is unital).
- consider only actions on **compact** quantum spaces.



Classically correspond to classical homogeneous spaces

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#### DEFINITION

 $\mathbb{G}, \mathbb{H} - l.c.q.g.$ 's.

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PIOTR M. SOŁTAN (WARSAW)

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  - $\textcircled{0}~\mathbb{H}$  is a closed quantum subgroup of  $\mathbb{G}$  in the sense of Vaes

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$$\widehat{\pi}\colon L^{\infty}(\widehat{\mathbb{H}})\longrightarrow L^{\infty}(\widehat{\mathbb{G}})$$

intertwining comultiplications.

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$$\left(\mathbb{H}_{\operatorname{Vaes}}\mathbb{G}\right) \Longrightarrow \left(\mathbb{H}_{\operatorname{SLW}}\mathbb{G}\right),$$

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$$\bullet \ \left( \mathbb{H}_{\underset{\mathrm{Vaes}}{\subset}} \mathbb{G} \right) \Longrightarrow \left( \mathbb{H}_{\underset{\mathrm{SLW}}{\subset}} \mathbb{G} \right),$$

• converse unclear, true in many cases.

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JULY 10, 2013 12 / 20 QUANTUM HOMOGENEOUS SPACES

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Image: A matrix and a matrix

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 $\bullet$  Define a quantum space  $\mathbb X$  setting

$$L^{\infty}(\mathbb{X}) = \big\{ \mathbf{x} \in L^{\infty}(\mathbb{G}) \, \big| \, \alpha(\mathbf{x}) = \mathbf{x} \otimes \mathbb{1} \big\}.$$

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- X is by definition the quotient space  $\mathbb{G}/\mathbb{H}$ .
- $L^{\infty}(\mathbb{X})$  is a left coideal in  $L^{\infty}(\mathbb{G})$ , i.e. an **embeddable** W<sup>\*</sup>-quantum G-space.

# $\mbox{Embeddable $W^*$-quantum $\mathbb{G}$-spaces}$

PIOTR M. SOŁTAN (WARSAW)

QUANTUM HOMOGENEOUS SPACES

JULY 10, 2013 13 / 20

#### DEFINITION

A quantum space  $\mathbb{X}$  is an **embeddable**  $W^*$ -**quantum**  $\mathbb{G}$ -space if  $L^{\infty}(\mathbb{X}) \subset L^{\infty}(\mathbb{G})$  and  $\Delta_{\mathbb{G}}(L^{\infty}(\mathbb{X})) \subset L^{\infty}(\mathbb{G}) \otimes L^{\infty}(\mathbb{X})$ .

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• 
$$\gamma: L^{\infty}(\mathbb{X}) \to L^{\infty}(\mathbb{G})$$
 — such that  $(\mathrm{id} \otimes \gamma) \circ \alpha = \Delta \circ \gamma$ 

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### THEOREM (CO-DUAL)

Define  $\widetilde{\mathbb{X}}$  by setting

$$L^{\infty}(\widetilde{\mathbb{X}}) = \left\{ y \in L^{\infty}(\widehat{\mathbb{G}}) \, \middle| \, \forall \, x \in L^{\infty}(\mathbb{X}) \, xy = yx \right\} = L^{\infty}(\mathbb{X})' \cap L^{\infty}(\widehat{\mathbb{G}}).$$

Then  $\widetilde{\mathbb{X}}$  is an embeddable  $W^*$  -quantum  $\widehat{\mathbb{G}}$  -space.

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# $\widetilde{\mathbb{X}}$ is a $W^*\text{-}\mathsf{guantum}\ \widehat{\mathbb{G}}\text{-}\mathsf{space}$

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# $\widetilde{\mathbb{X}}$ is a $W^*\text{-}\mathsf{guantum}\ \widehat{\mathbb{G}}\text{-}\mathsf{space}$

PROOF.

Take

•  $y \in L^{\infty}(\widetilde{\mathbb{X}})$ ,

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\widetilde{\mathbb{X}} is a W^*\text{-}\mathsf{guantum}\ \widehat{\mathbb{G}}\text{-}\mathsf{space}
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#### Take

- $y \in L^{\infty}(\widetilde{\mathbb{X}})$ ,
- $x \in L^{\infty}(\mathbb{X})$ .

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PROOF. Take •  $y \in L^{\infty}(\widetilde{X})$ , •  $x \in L^{\infty}(X)$ . Then

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PROOF. Take •  $y \in L^{\infty}(\widetilde{\mathbb{X}}),$ •  $x \in L^{\infty}(\mathbb{X}).$ Then  $\Delta_{\widehat{\mathbb{G}}}(y)(\mathbb{1} \otimes x) = \widehat{W}(y \otimes \mathbb{1})\widehat{W}^*(\mathbb{1} \otimes x)$  $= \Sigma W^* \Sigma(y \otimes \mathbb{1})\Sigma W \Sigma(\mathbb{1} \otimes x)$ 

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- $= \quad \Sigma W^* \Sigma (y \otimes 1) \Sigma W \Sigma (1 \otimes x) \Sigma W^* W \Sigma$
- $= \quad \Sigma W^*(\mathbb{1} \otimes y) W(x \otimes \mathbb{1}) W^* W \Sigma$

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- $= (\mathbb{1} \otimes \boldsymbol{x}) \Sigma \boldsymbol{W}^* (\mathbb{1} \otimes \boldsymbol{y}) \boldsymbol{W} \Sigma$

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THEOREM

*The co-dual of*  $\widetilde{\mathbb{X}}$  *is equal to*  $\mathbb{X}$ *.* 

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Image: A matrix and a matrix

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The co-dual of  $\mathbb{X}$  is equal to  $\mathbb{X}$ .

• The proof uses duality for crossed products by l.c.q.g.-actions (Vaes).

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- For  $\mathbb{X} = \mathbb{G}$ , we have  $\widetilde{\mathbb{X}} = \text{point } (L^{\infty}(\widetilde{\mathbb{X}}) = \mathbb{C}\mathbb{1}_{L^{\infty}(\widehat{\mathbb{G}})}).$

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#### THEOREM

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 $\mathbb{X}$  is of quotient type iff there exists a closed quantum subgroup  $\mathbb{H}$  of  $\mathbb{G}$  such that  $L^{\infty}(\widetilde{\mathbb{X}})$  is the image of  $L^{\infty}(\widehat{\mathbb{H}})$  in  $L^{\infty}(\widehat{\mathbb{G}})$ .

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#### DEFINITION

•  $\mathbb{G}$  — locally compact quantum group,

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•  $\mathrm{C}_0(\mathbb{X})$  is strongly dense in  $L^\infty(\mathbb{X})$ ,

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- $\mathbb{X}$  embeddable W<sup>\*</sup>-quantum  $\mathbb{G}$ -space,  $(L^{\infty}(\mathbb{X}) \subset L^{\infty}(\mathbb{G}))$

 $\mathbb X$  is an embeddable quantum homogeneous space if there is a C\*-subalgebra

 $\mathrm{C}_0(\mathbb{X}) \subset L^\infty(\mathbb{X})$ 

such that

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PIOTR M. SOŁTAN (WARSAW)

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PIOTR M. SOŁTAN (WARSAW)

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# WHAT?!

PIOTR M. SOŁTAN (WARSAW) QUANTUM HOMOGENEOUS SPACES

JULY 10, 2013 17 / 20

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QUANTUM HOMOGENEOUS SPACES

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- For classical groups, embeddable quantum homogeneous spaces correspond to homogeneous spaces.

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PIOTR M. SOLTAN (WARSAW)

QUANTUM HOMOGENEOUS SPACES

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PIOTR M. SOŁTAN (WARSAW) QUANTUM HOMOGENEOUS SPACES JULY

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QUANTUM HOMOGENEOUS SPACES

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#### THEOREM

If  $\mathbb X$  is of quotient type then  $\mathbb G$  is a classical locally compact group.

• In particular we find that **quantum** groups do not have diagonal subgroups.

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