Synchronous Games and Quantum Families of Maps

Quantum Group Seminar

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July 21, 2020, 4pm GMT

THEOREM

The assignment to a locally compact topological space X of the C^**-algebra* $\mathrm{C}_0(X)$ *defines an anti-equivalence of categories between*

- *the category of commutative* C^**-algebras with morphisms of* C^**-algebras*

and

- *the category of locally compact topological spaces with continuous maps.*

- A locally compact space is by definition Hausdorff.
- The "inverse" functor is defined as the assignment to a commutative C^*-algebra A its spectrum $\widehat{\mathsf{A}}$.
- A morphism of C^*-algebras from A to B is a $*$-homomorphism $\Phi:\mathsf{A}\to\mathrm{M}(\mathsf{B})$ such that $\overline{\Phi(\mathsf{A})\mathsf{B}}=\mathsf{B}$.

L.C. Top. Sp.	**Commutative C^*-algs.**
X	$\mathrm{C}_0(X)$
$\varphi : X \to Y$	$\Phi \in \mathrm{Mor}\big(\mathrm{C}_0(Y)\,\mathrm{C}_0(X)\big)$
X – compact	$\mathrm{C}_0(X)$ – unital
X – finite	$\mathrm{C}_0(X)$ – finite-dimensional
X – metrizable	$\mathrm{C}_0(X)$ – separable
probab. measure on X	state on $\mathrm{C}_0(X)$
$X \times Y$	$\mathrm{C}_0(X) \otimes \mathrm{C}_0(Y)$

- Note: $\mathrm{M}\big(\mathrm{C}_0(X)\big) = \mathrm{C}_{\mathrm{b}}(X)$.

DEFINITION

A **Quantum space** is an object of the category dual to the category of C*-algebras.

- A theorem about quantum spaces is nothing else than a theorem about C*-algebras.
- A quantum space $\mathcal{X}$ is called **compact** if the corresponding C*-algebra $C_0(\mathcal{X})$ is unital (in this case we write $C(\mathcal{X})$).
- Similarly, **finite** quantum spaces correspond to finite-dimensional C*-algebras.
- Classical (ordinary) locally compact spaces are particular examples of quantum spaces.

THEOREM (JAMES R. JACKSON, 1952)

Let X, Y and Z be topological spaces such that X is Hausdorff and Z is locally compact. Then the assignment to any $\psi \in \mathrm{C}(X \times Z, Y)$ of the map

$$Z \ni z \longmapsto \psi(\,\cdot\,, z) \in \mathrm{C}(X, Y)$$

is a homeomorphism of $\mathrm{C}(X \times Z, Y)$ onto $\mathrm{C}\big(Z, \mathrm{C}(X, Y)\big)$ with all three spaces of maps topologized by their respective compact-open topologies.

- Assume that X, Y and Z are locally compact. Then a continuous family of continuous maps from X to Y indexed by Z, i.e. a continuous map from Z to $\mathrm{C}(X, Y)$ is the same thing as an element of

$$\mathrm{Mor}\big(\mathrm{C}_0(Y), \mathrm{C}_0(X) \otimes \mathrm{C}_0(Z)\big).$$

DEFINITION

Let $\mathcal{X},\mathcal{Y}$ and $\mathcal{Z}$ be quantum spaces. A **quantum family of maps** from $\mathcal{X}$ to $\mathcal{Y}$ indexed by $\mathcal{Z}$ is an element

$$\Phi \in \mathrm{Mor}\big(\mathrm{C}_0(\mathcal{Y}), \mathrm{C}_0(\mathcal{X}) \otimes \mathrm{C}_0(\mathcal{Z})\big).$$

- A quantum family of maps is a very general object.
- Consequently interesting quantum families of maps must have additional features.
- How about a quantum version of the space $\mathrm{C}(X,Y)$ of all continuous maps from X to Y?

DEFINITION

Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be quantum spaces and let $\Phi \in \mathrm{Mor}\bigl(\mathrm{C}_0(\mathcal{Y}), \mathrm{C}_0(\mathcal{X}) \otimes \mathrm{C}_0(\mathcal{Z})\bigr)$ be a quantum family of maps.
We say that

- $\mathcal{Z}$ is the **quantum space of all maps** from $\mathcal{X}$ to $\mathcal{Y}$

and

- Φ is the **quantum family of all maps** from $\mathcal{X}$ to $\mathcal{Y}$

if for any quantum space $\mathcal{Z}'$ and any quantum family $\Psi \in \mathrm{Mor}\bigl(\mathrm{C}_0(\mathcal{Y}), \mathrm{C}_0(\mathcal{X}) \otimes \mathrm{C}_0(\mathcal{Z}')\bigr)$ there exists a unique $\Lambda \in \mathrm{Mor}\bigl(\mathrm{C}_0(\mathcal{Z}), \mathrm{C}_0(\mathcal{Z}')\bigr)$ such that

$$\begin{array}{ccc}
\mathrm{C}_0(\mathcal{Y}) & \xrightarrow{\ \Phi\ } & \mathrm{C}_0(\mathcal{X}) \otimes \mathrm{C}_0(\mathcal{Z}) \\
\Big\| & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\Lambda} \\
\mathrm{C}_0(\mathcal{Y}) & \xrightarrow{\ \Psi\ } & \mathrm{C}_0(\mathcal{X}) \otimes \mathrm{C}_0(\mathcal{Z}')
\end{array}$$

- Let X, Y and Z be locally compact spaces. For the quantum space of all maps from X to Y to exist it is necessary that $\mathrm{C}(X, Y)$ be locally compact in the compact-open topology.
- This will certainly be the case when X is finite and Y is compact.

THEOREM (S.L. WORONOWICZ 1979, P.S. 2009)

Let $\mathcal{X}$ be a finite quantum space and let $\mathcal{Y}$ be a compact quantum space such that $\mathrm{C}(\mathcal{Y})$ is finitely generated. Then the quantum space of all maps from $\mathcal{X}$ to $\mathcal{Y}$ exists and it is compact.

- Let $\mathcal{Z}$ be the quantum space of all maps $\mathcal{X} \to \mathcal{Y}$.
- The universal family

$$\mathbf{\Phi} \in \mathrm{Mor}\big(\mathrm{C}(\mathcal{Y}), \mathrm{C}(\mathcal{X}) \otimes \mathrm{C}(\mathcal{Z})\big)$$

corresponds to the evaluation mapping

$$X \times \mathrm{C}(X, Y) \ni (x, \psi) \longmapsto \psi(x) \in Y$$

for classical spaces X and Y.

Example

- Consider $\mathcal{X}$ such that $\mathrm{C}(\mathcal{X}) = M_2$.
- Take $\mathcal{Y} = \{\bullet, \bullet\}$, i.e. $\mathrm{C}(\mathcal{Y}) = \mathbb{C}^2$.
- Let $\mathcal{Z}$ be the quantum space of all maps $\mathcal{X} \to \mathcal{Y}$.
- Then $\mathrm{C}(\mathcal{Z})$ is the universal unital C*-algebra generated by three elements p, q and z with relations

$$\begin{aligned} p &= p^*, & p &= p^2 + z^*z, & zp &= (\mathbb{1} - q)z, \\ q &= q^*, & q &= q^2 + zz^*. \end{aligned}$$

Definition

Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{P}_1$ and $\mathcal{P}_2$ be quantum spaces and let

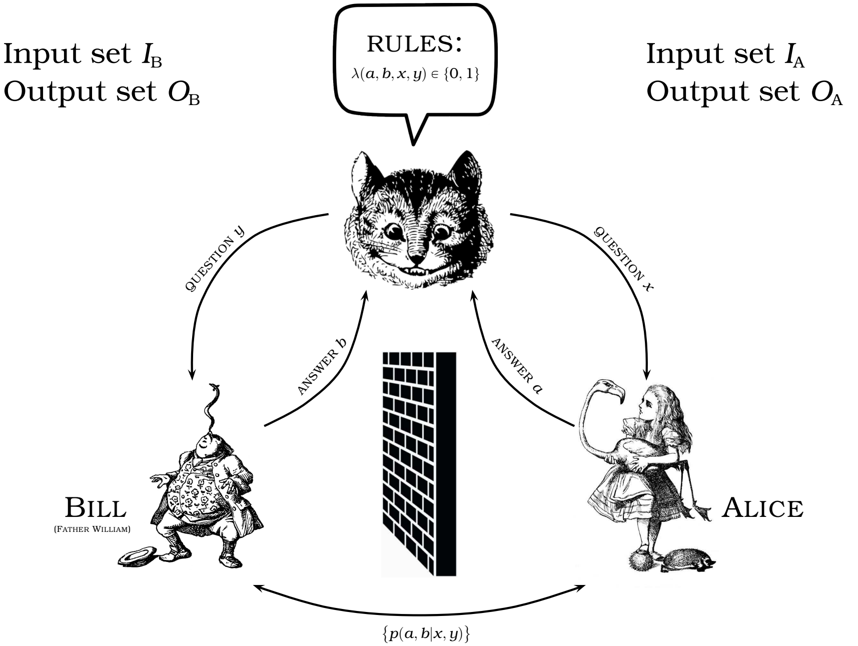
$$\Psi_{12} \in \operatorname{Mor}\big(\mathrm{C}(\mathcal{X}_2), \mathrm{C}(\mathcal{X}_1) \otimes \mathrm{C}(\mathcal{P}_1)\big),$$
$$\Psi_{23} \in \operatorname{Mor}\big(\mathrm{C}(\mathcal{X}_3), \mathrm{C}(\mathcal{X}_2) \otimes \mathrm{C}(\mathcal{P}_2)\big)$$

be quantum families of maps. The **composition** of Ψ_{32} and Ψ_{21} is

$$\Psi_{12} \vartriangle \Psi_{23} = (\Psi_{12} \otimes \mathrm{id}) \circ \Psi_{23} \in \operatorname{Mor}\big(\mathrm{C}(\mathcal{X}_3), \mathrm{C}(\mathcal{X}_1) \otimes (\mathrm{C}(\mathcal{P}_2) \otimes \mathrm{C}(\mathcal{P}_2))\big).$$

- The composition of classical families (with classical parameter spaces) is exactly the family of all compositions of elements of both families.
- Composition is associative:

$$\big(\Psi_{12} \vartriangle \Psi_{23}\big) \vartriangle \Psi_{34} = \Psi_{12} \vartriangle \big(\Psi_{23} \vartriangle \Psi_{34}\big).$$


Input set I_B
Output set O_B
RULES:
$\lambda(a, b, x, y) \in \{0, 1\}$
Input set I_A
Output set O_A
QUESTION y
QUESTION x
ANSWER b
ANSWER a
BILL
(FATHER WILLIAM)
ALICE
$\{p(a, b|x, y)\}$

- The strategies $\{p(a,b|x,y)\}$ can be **classical**:

$$p(a,b|x,y)=\mathbb{P}\bigl(\{f_x=a,\ g_y=b\}\bigr),$$

where

$$\{f_x\colon\Omega\to O_{\mathrm{A}}\}_{x\in I_{\mathrm{A}}}\quad\text{and}\quad\{g_y\colon\Omega\to O_{\mathrm{A}}\}_{y\in I_{\mathrm{B}}}$$

are random variables on a probability space $(\Omega,\mathbb{P})$.

- They can be **quantum**

$$p(a,b|x,y)=\bigl\langle\Psi\bigl|E_{x,a}\otimes F_{y,b}\bigr|\Psi\bigr\rangle,$$

where $\Psi\in\mathscr{H}_{\mathrm{A}}\otimes\mathscr{H}_{\mathrm{B}}$ is a state and

$$\{E_{x,a}\}_{(x,a)\in I_{\mathrm{A}}\times O_{\mathrm{A}}}\quad\text{and}\quad\{F_{y,b}\}_{(y,b)\in I_{\mathrm{B}}\times O_{\mathrm{B}}}$$

are **quantum measurements**:

$$\forall\,x,y\quad\sum_{a\in O_{\mathrm{A}}}E_{x,a}=\mathbb{1},\quad\sum_{b\in O_{\mathrm{B}}}F_{y,b}=\mathbb{1}.$$

- A game is **synchronous** if $I_{\mathrm{A}} = I_{\mathrm{B}} = I$, $O_{\mathrm{A}} = O_{\mathrm{B}} = O$ and

$$\Bigl(\, a \neq b \,\Bigr) \implies \Bigl(\, \lambda(a,b,x,x) = 0 \,\Bigr)$$

  for all $x \in I$.

- The **quantum space of strategies** $\mathcal{P}$ of the game is defined by setting $\mathrm{C}(\mathcal{P})$ to be the universal C*-algebra generated by projections

$$\bigl\{p_{x,a}\bigr\}_{(x,a)\in I\times O}$$

  such that $\sum\limits_{a\in O} p_{x,a} = \mathbb{1}$ for all $x \in I$ and

$$p_{x,a}p_{y,b} = \lambda(a,b,x,y)p_{x,a}p_{y,b}.$$

  for all $x, y \in I$ and $a, b \in O$.

- We fix a synchronous game (I, O, λ).

PROPOSITION

There exists a unique $\Phi \in \operatorname{Mor}\bigl(\mathbb{C}^O, \mathbb{C}^I \otimes \mathrm{C}(\mathcal{P})\bigr)$ *such that*

$$\Phi(e_a) = \sum_{x \in I} e_x \otimes p_{x,a}, \qquad a \in O,$$

where $\{e_a\}_{a \in O}$ *and* $\{e_x\}_{x \in I}$ *are standard basis of* $\mathbb{C}^I$ *and* $\mathbb{C}^O$*.*

- Thus there is a natural quantum family of maps from I to O indexed by the quantum space $\mathcal{P}$.

THEOREM

For any quantum space $\mathcal{X}$ *and any quantum family* $\Psi \in \mathrm{Mor}\big(\mathbb{C}^O, \mathbb{C}^I \otimes \mathrm{C}_0(\mathcal{X})\big)$ *such that*

$$(\delta_x \otimes \delta_y \otimes \mathrm{id})\big(\Psi(e_a)_{13}\Psi(e_b)_{23}\big) = \lambda(a,b,x,y)\cdot(\delta_x \otimes \delta_y \otimes \mathrm{id})\big(\Psi(e_a)_{13}\Psi(e_b)_{23}\big)$$

for all $x, y \in I$ *and* $a, b \in O$ *there exists a unique*

$$\Theta \in \mathrm{Mor}\big(\mathrm{C}(\mathcal{P}), \mathrm{C}_0(\mathcal{X})\big)$$

such that

$$\begin{array}{ccc} \mathbb{C}^O & \xrightarrow{\quad\Phi\quad} & \mathbb{C}^I \otimes \mathrm{C}(\mathcal{P}) \\ \| & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\Theta} \\ \mathbb{C}^O & \xrightarrow{\quad\Psi\quad} & \mathbb{C}^I \otimes \mathrm{C}_0(\mathcal{X}) \end{array}$$

- Consider now a game with $I = O$ and such that

$$\Big(\lambda(i,j,k,l) = 0\Big) \implies \Big(\forall r,s \in I \quad \lambda(i,j,r,s)\lambda(r,s,k,l) = 0\Big) \quad (\star)$$

Theorem

For a synchronous game with $O = I$ and rules satisfying condition $(\star)$

1. *there exists a unique* $\Delta \in \mathrm{Mor}(\mathrm{C}(\mathcal{P}), \mathrm{C}(\mathcal{P}) \otimes \mathrm{C}(\mathcal{P}))$ *such that*

$$(\Phi \otimes \mathrm{id}) \circ \Phi = (\mathrm{id} \otimes \Delta) \circ \Phi,$$

2. Δ *is coassociative:* $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ *and consequently endows* $\mathcal{P}$ *with the structure of a compact quantum semigroup; moreover* Φ *is an action of this quantum semigroup on* I.

- Consider a synchronous game with $I = O$ and condition ($\star$).
- The fact that $\Phi \in \mathrm{Mor}\big(\mathbb{C}^I, \mathbb{C}^I \otimes \mathrm{C}(\mathcal{P})\big)$ **preserves the uniform measure** μ on I:

$$(\mu \otimes \mathrm{id})\Phi(a) = \mu(a)\mathbb{1}, \qquad a \in \mathbb{C}^I$$

  translates to $\sum_{i\in I} p_{i,j} = \mathbb{1}$ for all $j \in I$.
- Define $\mathrm{C}(\widetilde{\mathcal{P}})$ as the universal C*-algebra generated by projections $\{\widetilde{p}_{i,j}\}_{i,j\in I}$ such that
  - $p_{k,i}p_{l,j} = \lambda(i,j,k,l)p_{k,i}p_{l,j}$ for all $i,j,k,l \in I$,
  - $\sum_{j\in I} p_{i,j} = \mathbb{1}$ for all $i \in I$, $\sum_{i\in I} p_{i,j} = \mathbb{1}$ for all $j \in I$.
- There exists a unique quantum family of maps $\widetilde{\Phi} \in \mathrm{Mor}\big(\mathbb{C}^I, \mathbb{C}^I \otimes \mathrm{C}(\widetilde{\mathcal{P}})\big)$ such that

$$\widetilde{\Phi}(e_j) = \sum_{i\in I} e_{i,j} \otimes \widetilde{p}_{i,j}, \qquad j \in I.$$

THEOREM

In the situation described on the previous slide

1. *there exits a unique* $\widetilde{\Delta} \in \mathrm{Mor}\big(\mathrm{C}(\widetilde{\mathcal{P}}, \mathrm{C}(\widetilde{\mathcal{P}} \otimes \mathrm{C}(\widetilde{\mathcal{P}})$ *such that*

$$\widetilde{\Phi} \vartriangle \widetilde{\Phi} = (\mathbf{id} \otimes \widetilde{\Delta}) \circ \widetilde{\Phi},$$

2. $\widetilde{\Delta}$ *is coassociative,*
3. $\widetilde{\mathcal{P}}$ *is a compact quantum group.*

- When the game is the **graph endomorphism game** of a finite graph $\mathscr{G}$, the quantum group $\widetilde{\mathcal{P}}$ is the **quantum automorphism group** of $\mathscr{G}$ defined by T. Banica.

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Thank you for your attention.