On some invariants of quantum groups

Quantum Groups and interactions

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May 24, 2023

Motivation

- In recent work with J. Krajczok we looked at examples of compact quantum groups $\mathbb{G}$ such that $\mathsf{L}^{\infty}(\mathbb{G})$ is an injective factor of type III.
- We constructed many such examples by taking $\mathbb{G}$ of the form

$$\mathbb{G} = \bigtimes_{n=1}^{\infty} \mathbb{H}_{\nu_n, q_n},$$

where $\{\mathbb{H}_{\nu_n,q_n}\}_{n\in\mathbb{N}}$ is a sequence of compact quantum groups defined as

$$\mathbb{H}_{\nu_n,q_n} = \mathbb{Q} \bowtie \mathrm{SU}_{q_n}(2)$$

with the action of $r \in \mathbb{Q}$ by the automorphism $\tau^{\mathrm{SU}_{q_n}(2)}_{\nu_n r}$.
- We always assume that $\nu_n \log|q_n| \notin \pi\mathbb{Q}$.

Motivation

- With appropriate choice of $\{(\nu_n, q_n)\}_{n\in\mathbb{N}}$ we obtained examples with $\mathsf{L}^\infty(\mathbb{G})$ injective and
  - of type III_λ for $\lambda \in]0,1[$,
  - of type III_1,
  - of type III_0 (uncountably many pairwise non-isomorphic factors).
- Next we aimed at constructing families of pairwise non-isomorphic quantum groups sharing the same injective factor.
- In order to distinguish between our examples we introduced some invariants.

The invariants

Definition

Let $\mathbb{G}$ be a locally compact quantum group. We define

$$\begin{aligned} T^{\tau}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\tau^{\mathbb{G}}_t=\mathrm{id}\bigr\},\\ T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\tau^{\mathbb{G}}_t\in\mathrm{Inn}(\mathsf{L}^\infty(\mathbb{G}))\bigr\},\\ T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\tau^{\mathbb{G}}_t\in\overline{\mathrm{Inn}}(\mathsf{L}^\infty(\mathbb{G}))\bigr\}, \end{aligned}$$

where $\overline{\mathrm{Inn}}(\cdot)$ denotes the approximately inner automorphisms.

- If $\mathbb{G}$ and $\mathbb{H}$ are isomorphic then $T^{\tau}(\mathbb{G})=T^{\tau}(\mathbb{H})$.
- $T^{\tau}(\mathbb{G})\subset T^{\tau}_{\mathrm{Inn}}(\mathbb{G})\subset T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G})$ and each invariant is a subgroup of $\mathbb{R}$.
- We have $T^{\tau}(\mathbb{G})=T^{\tau}(\widehat{\mathbb{G}})$.
- A compact $\mathbb{G}$ is of Kac type if and only of $T^{\tau}(\mathbb{G})=\mathbb{R}$.

Examples

- $T^{\tau}\big(\mathrm{SU}_q(2)\big) = T^{\tau}_{\mathrm{Inn}}\big(\mathrm{SU}_q(2)\big) = T^{\tau}_{\overline{\mathrm{Inn}}}\big(\mathrm{SU}_q(2)\big) = \frac{\pi}{\log|q|}\mathbb{Z}$.
- $T^{\tau}(\mathbb{H}_{\nu,q}) = \frac{\pi}{\log|q|}\mathbb{Z}$, $T^{\tau}_{\mathrm{Inn}}(\mathbb{H}_{\nu,q}) = \nu\mathbb{Q} + \frac{\pi}{\log|q|}\mathbb{Z}$, $T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{H}_{\nu,q}) = \mathbb{R}$.

More examples

Now let $\mathbb{G} = \bigtimes_{n=1}^{\infty} \mathbb{H}_{\nu_n,q_n}$.

- If $(\nu_n, q_n) = (\nu, q)$ for all n then $\mathsf{L}^{\infty}(\mathbb{G})$ is the injective factor of type III_{q^2} and $T^{\tau}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \frac{\pi}{\log|q|}\mathbb{Z}$, $T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = \mathbb{R}$.
- One can choose $\big((\nu_n, q_n)\big)_{n\in\mathbb{N}}$ so that $\mathsf{L}^{\infty}(\mathbb{G})$ is the injective factor of type III_1 and $T^{\tau}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \{0\}$, $T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = \mathbb{R}$.
- One can choose $\big((\nu_n, q_n)\big)_{n\in\mathbb{N}}$ so that $\mathsf{L}^{\infty}(\mathbb{G})$ an injective factor of type III_0 (can get uncountably many of them) and $T^{\tau}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{Z}$, $T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = \mathbb{R}$.

Application of the invariants

- We constructed examples of compact quantum groups $\mathbb{G}$ of the form $\overset{\infty}{\underset{n=1}{\times}} \mathbb{H}_n$ such that
  - for any countable subgroup $\Gamma \subset \mathbb{R}$ (taken with discrete topology),
  - with $\mathbb{K} = \Gamma \ltimes \mathbb{G}$ (action by the scaling automorphisms)

  we have
  - $T^\tau(\mathbb{K}) = T^\tau(\mathbb{G}) = T^\tau_{\mathrm{Inn}}(\mathbb{G}) = \bigcap_{n=1}^{\infty} T^\tau(\mathbb{H}_n)$,
  - $T^\tau_{\mathrm{Inn}}(\mathbb{K}) = \Gamma + \bigcap_{n=1}^{\infty} T^\tau(\mathbb{H}_n)$,
  - $T^\tau_{\overline{\mathrm{Inn}}}(\mathbb{K}) = \mathbb{R}$.
- In this construction we can have $\mathsf{L}^\infty(\mathbb{K})$ isomorphic to the injective factor of type III_λ $(0 < \lambda \leqslant 1)$ and $\bigcap_{n=1}^{\infty} T^\tau(\mathbb{H}_n) = \{0\}$.
- There are uncountably many countable subgroups $\Gamma \subset \mathbb{R}$.

More invariants

Definition

Let $\mathbb{G}$ be a locally compact quantum group with left Haar measure $\boldsymbol{h}$. In addition to $T^{\tau}(\mathbb{G})$, $T^{\tau}_{\mathrm{Inn}}(\mathbb{G})$ and $T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G})$ we define

$$\begin{aligned} T^{\sigma}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\sigma^{\boldsymbol{h}}_t=\mathrm{id}\bigr\},\\ T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\sigma^{\boldsymbol{h}}_t\in\mathrm{Inn}(\mathsf{L}^{\infty}(\mathbb{G}))\bigr\},\\ T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\sigma^{\boldsymbol{h}}_t\in\overline{\mathrm{Inn}}(\mathsf{L}^{\infty}(\mathbb{G}))\bigr\},\\ \mathrm{Mod}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\delta^{\mathrm{i}t}=\mathbb{1}\bigr\}, \end{aligned}$$

where δ is the modular element of $\mathbb{G}$.

- Clearly $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G})=T(\mathsf{L}^{\infty}(\mathbb{G}))$.
- $T^{\circ}_{\bullet}(\mathbb{X})$, $\mathrm{Mod}(\mathbb{X})$ with $\circ\in\{\tau,\sigma\}$, $\bullet\in\{\ ,\mathrm{Inn},\overline{\mathrm{Inn}}\}$ and $\mathbb{X}\in\{\mathbb{G},\widehat{\mathbb{G}}\}$ yield 14 invariants.

Proposition

For any locally compact quantum group $\mathbb{G}$ we have

$$\begin{aligned} T^{\sigma}(\mathbb{G}) &= T^{\tau}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}),\\ T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}) &= T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}),\\ T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}) &= T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}),\\ \operatorname{Mod}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}) &\subset \tfrac{1}{2}\, T^{\tau}(\mathbb{G}). \end{aligned}$$

- The first equality above together with $T^{\tau}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}})$ reduces the list to 11 (invariants $T^{\sigma}(\mathbb{G})$, $T^{\sigma}(\widehat{\mathbb{G}})$ and $T^{\tau}(\widehat{\mathbb{G}})$ are determined by the remaining ones).
- If $\mathbb{G}$ is compact then $\operatorname{Mod}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathbb{R}$.
- If additionally $\mathsf{L}^{\infty}(\mathbb{G})$ is semifinite then $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = \mathbb{R}$.

- The invariants can be calculated for a number of well-known quantum groups.

Example

With $\mathbb{G}=\mathrm{E}_q(2)$ for some $q\in\,]0,1[$ we have

$$T^\tau(\mathbb{G})=T^\tau_{\mathrm{Inn}}(\mathbb{G})=T^\tau_{\overline{\mathrm{Inn}}}(\mathbb{G})=T^\sigma(\mathbb{G})=T^\tau(\widehat{\mathbb{G}})=T^\sigma(\widehat{\mathbb{G}})=\mathrm{Mod}(\widehat{\mathbb{G}})=\tfrac{\pi}{\log q}\mathbb{Z}$$
$$T^\sigma_{\mathrm{Inn}}(\mathbb{G})=T^\sigma_{\overline{\mathrm{Inn}}}(\mathbb{G})=T^\tau_{\mathrm{Inn}}(\widehat{\mathbb{G}})=T^\tau_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}})=T^\sigma_{\mathrm{Inn}}(\widehat{\mathbb{G}})=T^\sigma_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}})=\mathrm{Mod}(\mathbb{G})=\mathbb{R}.$$

- The invariants can also be calculated for q-deformations of compact semisimple Lie groups. In particular

$$T^\tau_{\mathrm{Inn}}\big(\mathrm{SU}_q(3)\big)=\tfrac{\pi}{2\log q}\mathbb{Z}\quad\text{and}\quad T^\tau\big(\mathrm{SU}_q(3)\big)=\tfrac{\pi}{\log q}\mathbb{Z}.$$

Can all scaling automorphisms be inner?

"Conjecture"

Let $\mathbb{G}$ be a compact quantum group such that $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$. Then $\mathbb{G}$ is of Kac type.

- Suppose we know that $\mathbb{G}$ is not of Kac type. Then the validity of the "conjecture" for $\mathbb{G}$ reduces to whether $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$ or not.

Theorem (Jacek Krajczok + P.M.S.)

Let $\mathbb{G}$ be a compact quantum group with a two-dimensional representation U such that $2 = \dim U < \dim_{\mathrm{q}} U$. Then $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) \neq \mathbb{R}$.

- In what follows we will assume that $\mathbb{G}$ is a compact quantum group.
- If U is an irreducible representation of $\mathbb{G}$ we let ρ_U denote the associated invertible positive element of $\mathrm{Mor}(U, U^{\mathrm{cc}})$ such that $\mathrm{Tr}(\rho_U) = \mathrm{Tr}(\rho_U^{-1})$.
- Furthermore, let $\Gamma(U) = \max \mathrm{Sp}(\rho_U)$ and $\gamma(U) = \min \mathrm{Sp}(\rho_U)$.

Theorem

Let $\{U^n\}_{n\in\mathbb{N}}$ be a family of irreducible representations of $\mathbb{G}$ such that

- $\Gamma(U^n) \xrightarrow[n\to\infty]{} +\infty$,
- $\inf\limits_{n\in\mathbb{N}} \min\left\{\frac{1}{\gamma(U^n)\dim_{\mathrm{q}} U^n}, \frac{\Gamma(U^n)}{\dim_{\mathrm{q}} U^n}\right\} > 0$.

Then $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) \neq \mathbb{R}$.

Sketch of proof

1. Let us assume that $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$.
2. Then there is a strongly continuous group $\{v_t\}_{t\in\mathbb{R}}$ of unitaries in $\mathsf{L}^{\infty}(\mathbb{G})$ such that $\tau^{\mathbb{G}}_t = \mathrm{Ad}_{v_t}$ for all t.
3. We let $\varepsilon_t = \|v_t - \mathbb{1}\|_2$ (clearly $\varepsilon_t \xrightarrow[t\to 0]{} 0$).
4. We note that $v_t \in \mathsf{L}^{\infty}(\mathbb{G})^{\sigma}$.
5. Let $\mathbb{H} = \mathbb{G} \times \mathbb{G}$ and let $X_n = \bigl(U^n_{1,\dim U^n}\bigr) \otimes \bigl(\overline{U^n}_{\dim U^n,1}\bigr) \in \mathrm{Pol}(\mathbb{H})$.
6. Using the orthogonality relations (on $\mathbb{G}$) we find that

$$\|X_n\|_2^2 = \frac{1}{\gamma(U^n)\,\dim_{\mathrm{q}} U^n}\,\frac{\Gamma(U^n)}{\dim_{\mathrm{q}} U^n},$$

so that $\|X_n\|_2 \geqslant c$, where

$$c = \inf_{n\in\mathbb{N}} \min\biggl\{\frac{1}{\gamma(U^n)\,\dim_{\mathrm{q}} U^n}, \frac{\Gamma(U^n)}{\dim_{\mathrm{q}} U^n}\biggr\}.$$

Sketch of proof

7. We calculate that for all t we have $\sigma_t^{\boldsymbol{h}_{\mathbb{H}}}(X_n)=X_n$ and $\tau_t^{\mathbb{H}}(X_n)=\left(\frac{\gamma(U^n)}{\Gamma(U^n)}\right)^{2it}X_n$.
8. It follows that
$$\left|\left(\frac{\gamma(U^n)}{\Gamma(U^n)}\right)^{2it}-1\right|c\leqslant\left|\left(\frac{\gamma(U^n)}{\Gamma(U^n)}\right)^{2it}-1\right|\|X_n\|_2=\left\|\tau_t^{\mathbb{H}}(X_n)\Omega_{\mathbb{H}}-X_n\Omega_{\mathbb{H}}\right\|.$$
9. Next, using the fact that $\tau_t^{\mathbb{H}}=\mathrm{Ad}_{v_t\otimes v_t}$, $v_t\in\mathsf{L}^\infty(\mathbb{G})^\sigma$ and $X_n\in\mathsf{L}^\infty(\mathbb{H})^\sigma$ we arrive at the estimate
$$\left|\left(\frac{\gamma(U^n)}{\Gamma(U^n)}\right)^{2it}-1\right|c\leqslant 4\varepsilon_t.$$
10. Let $t_1>0$ be such that $\varepsilon_t<\frac{c}{4}$ for $t\in\,]0,t_1]$.
11. Furthermore, as $\Gamma(U^n)\xrightarrow[n\to\infty]{}+\infty$ and $\gamma(U^n)\leqslant 1$, there exists m such that
$$\log\left(\frac{\Gamma(U^m)}{\gamma(U^m)}\right)\geqslant\frac{\pi}{2t_1}.$$

Sketch of proof

12. It follows that $t_2 = \frac{\pi}{2}\log\Bigl(\frac{\Gamma(U^m)}{\gamma(U^m)}\Bigr)^{-1}$ belongs to $]0, t_1]$, i.e. $4\varepsilon_{t_2} < c$.
13. Finally note that

$$\left|\Bigl(\frac{\gamma(U^m)}{\Gamma(U^m)}\Bigr)^{2\mathrm{i}t_2} - 1\right| = 2,$$

so

$$2c = \left|\Bigl(\frac{\gamma(U^m)}{\Gamma(U^m)}\Bigr)^{2\mathrm{i}t_2} - 1\right| c \leqslant 4\varepsilon_{t_2} < c.$$

This contradiction shows that not all scaling automorphisms of $\mathbb{G}$ are inner.

Proposition

If $\mathbb{G}$ is a compact quantum group with a representation U such that $2 = \dim U < \dim_{\mathrm{q}} U$ then a sequence of irreducible representations $\{U^n\}_{n\in\mathbb{N}}$ as in the previous theorem can be constructed.

- Representations $\{U^n\}_{n\in\mathbb{N}}$ are constructed as subrepresentations of n-fold tensor products of copies of U and $\overline{U}$.
- The estimates on $\Gamma(U^n)$ and $\gamma(U^n)$ follow from properties of the fundamental representation of U_F^+ which maps "onto" U.

DUALS OF TYPE I DISCRETE QUANTUM GROUPS

THEOREM (JACEK KRAJCZOK + P.M.S.)

Let $\mathbb{\Gamma}$ be a second countable discrete quantum group of type I*. Then the "conjecture" holds for $\widehat{\mathbb{\Gamma}}$.*

- A locally compact quantum group $\mathbb{G}$ is **second countable** if $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})$ is a separable C*-algebra (equivalently either of $\mathrm{C}_0(\mathbb{G})$, $\mathsf{L}^1(\mathbb{G})$ or $\mathsf{L}^2(\mathbb{G})$ is separable).
- $\mathbb{G}$ is of **type** I if $\mathrm{C}_0^{\mathrm{u}}(\widehat{\mathbb{G}})$ is a type I C*-algebra.

REMARK

One cannot remove the assumption of $\mathbb{G}$ being second countable because taking bicrossed product by discretized $\mathbb{R}$ acting by scaling automorphisms one can construct non-Kac type compact quantum groups with all scaling automorphisms inner.

On the proof

- The Hilbert space $\mathsf{L}^2(\hat{\mathbb{\Gamma}})$ is decomposed as a direct integral $\int\limits_{\operatorname{Irr}\mathbb{\Gamma}}^{\oplus} \mathrm{HS}(\mathscr{H}_\pi)\,\mathrm{d}\mu(\pi)$ with $x \in \mathsf{L}^\infty(\hat{\mathbb{\Gamma}})$ acting as $\int\limits_{\operatorname{Irr}\mathbb{\Gamma}}^{\oplus} x_\pi \otimes \mathbb{1}_{\overline{\mathscr{H}_\pi}}\,\mathrm{d}\mu(\pi)$ (upon identification of $\mathrm{B}(\mathrm{HS}(\mathscr{H}_\pi)) = \mathrm{B}(\mathscr{H}_\pi) \mathbin{\bar{\otimes}} \mathrm{B}(\overline{\mathscr{H}_\pi})$).
- We have $\boldsymbol{h}_{\hat{\mathbb{\Gamma}}}(x) = \int\limits_{\operatorname{Irr}\mathbb{\Gamma}} \operatorname{Tr}(D_\pi^{-2} x_\pi)\,\mathrm{d}\mu(\pi)$ for a certain measurable field $\pi \mapsto D_\pi$ of non-singular positive self-adjoint operators.
- The proof relies on careful accounting of how elements of the direct integral decomposition of matrix elements of irreps of $\hat{\mathbb{\Gamma}}$ shift spectral subspaces of the operators D_π.
- The outcome is that for $\alpha \in \operatorname{Irr}\hat{\mathbb{\Gamma}}$ and $t \in T^\tau_{\mathrm{Inn}}(\hat{\mathbb{\Gamma}})$ the number $\Gamma(U^\alpha)^{2it}$ is a root of unity. This implies that if $\hat{\mathbb{\Gamma}}$ is not of Kac type then $T^\tau_{\mathrm{Inn}}(\hat{\mathbb{\Gamma}})$ is countable.

Thank you for your attention