Quantum groups and factors

Quantum groups, Hopf algebras and monoidal categories

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1. Compact quantum groups
   - Definition
   - Examples
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2. Factors from compact quantum groups
   - Type I
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A **compact quantum group** $\mathbb{G}$ is described by

- a von Neumann algebra $\mathsf{L}^\infty(\mathbb{G})$,
- a unital $*$-homomorphism $\Delta\colon \mathsf{L}^\infty(\mathbb{G}) \to \mathsf{L}^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} \mathsf{L}^\infty(\mathbb{G})$

(continuous in the σ-weak topology) such that

- $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$,
- there exists a faithful state $\boldsymbol{h}$ on $\mathsf{L}^\infty(\mathbb{G})$ such that

$$(\boldsymbol{h} \otimes \mathrm{id})\Delta(x) = \boldsymbol{h}(x)\mathbb{1} = (\mathrm{id} \otimes \boldsymbol{h})\Delta(x), \qquad x \in \mathsf{L}^\infty(\mathbb{G}).$$

(Haar measure)

Examples include:

- $\mathsf{L}^\infty(\mathbb{G}) = \mathsf{L}^\infty(G)$ for a compact group G, $\Delta(f)(x, y) = f(xy)$ for $f \in \mathsf{L}^\infty(G)$, $x, y \in G$ and

$$\boldsymbol{h}(f) = \int_G f \, \mathrm{d}h, \qquad f \in \mathsf{L}^\infty(G),$$

  where h is the Haar measure on G.

- $\mathsf{L}^\infty(\mathbb{G}) = L(\Gamma)$, i.e. the von Neumann algebra generated by the range of the left regular representation $\gamma \mapsto \lambda_\gamma$ of a discrete group Γ on $\ell_2(\Gamma)$, $\Delta(\lambda_\gamma) = \lambda_\gamma \otimes \lambda_\gamma$ for all $\gamma \in \Gamma$ and

$$\boldsymbol{h}(x) = \langle \delta_e | x \delta_e \rangle, \qquad x \in L(\Gamma),$$

  where δ_e is the “delta function” at $e \in \Gamma$.

- Let $\mathbb{G}$ be a compact quantum group.
  - There exist
    - a set $\operatorname{Irr}\mathbb{G}$,
    - a family of finite dimensional Hilbert spaces $\{\mathsf{H}^\alpha\}_{\alpha\in\operatorname{Irr}\mathbb{G}}$,
    - unitary elements $U^\alpha \in \mathrm{B}(\mathsf{H}^\alpha)\otimes \mathsf{L}^\infty(\mathbb{G})$,
    - a choice of an orthonormal basis $\{\xi^\alpha_1,\dots,\xi^\alpha_{n_\alpha}\}$ in each H^α

    such that the corresponding matrix elements $U^\alpha_{i,j}$ of all U^α span a dense unital $*$-subalgebra of $\mathsf{L}^\infty(\mathbb{G})$ and satisfy

$$\boldsymbol{h}\big(U^{\alpha}_{i,j}{}^{*}U^{\alpha}_{k,l}\big)=\frac{\delta_{k,i}\delta_{j,l}\rho_{\alpha,j}^{-1}}{M_\alpha},\qquad \boldsymbol{h}\big(U^{\alpha}_{k,l}U^{\alpha}_{i,j}{}^{*}\big)=\frac{\delta_{k,i}\delta_{j,l}\rho_{\alpha,j}}{M_\alpha},$$

    where $\rho_{\alpha,1}\geqslant\dots\geqslant\rho_{\alpha,n_\alpha}>0$ and $M_\alpha=\sum\limits_i\rho_{\alpha,i}$.

- Let $\mathbb{G}$ be a compact quantum group.
  - There exists a one-parameter group $(\tau_t^{\mathbb{G}})_{t\in\mathbb{R}}$ of automorphisms of $L^\infty(\mathbb{G})$ such that

$$\tau_t^{\mathbb{G}}(U_{i,j}^{\alpha}) = \rho_{\alpha,i}^{\mathrm{i}t} U_{i,j}^{\alpha} \rho_{\alpha,j}^{-\mathrm{i}t}$$

    for all i, j, α and t. **(scaling group)**
  - The modular group $(\sigma_t^{\boldsymbol{h}})_{t\in\mathbb{R}}$ of $\boldsymbol{h}$ acts on $U_{i,j}^{\alpha}$ as follows:

$$\sigma_t^{\boldsymbol{h}}(U_{i,j}^{\alpha}) = \rho_{\alpha,i}^{\mathrm{i}t} U_{i,j}^{\alpha} \rho_{\alpha,j}^{\mathrm{i}t}$$

    for all i, j, α and t.
  - The two groups of automorphisms commute.

Theorem (Jacek Krajczok & P.M.S.)

Let H be a Hilbert space with $\dim\mathsf{H} > 1$. Then there does not exist a compact quantum group $\mathbb{G}$ such that $\mathsf{L}^{\infty}(\mathbb{G}) \cong \mathrm{B}(\mathsf{H})$.

- $\dim\mathsf{H} < +\infty$ is easy because then $\mathsf{L}^{\infty}(\mathbb{G})$ cannot be simple.
- $\dim\mathsf{H} > \aleph_0$ cannot happen because there are no faithful states on $\mathrm{B}(\mathsf{H})$ for non-separable H.
- Thus the only non-trivial case is that of an infinite dimensional separable Hilbert space H.
- One can tweak the proof to show that there is no compact quantum group $\mathbb{G}$ such that $\mathsf{L}^{\infty}(\mathbb{G}) \cong \mathsf{N} \oplus \mathrm{B}(\mathsf{H})$ for any von Neumann algebra N.

Step 1.

- Suppose $\mathbb{G}$ is a compact quantum group with $\mathsf{L}^\infty(\mathbb{G}) \cong \mathrm{B}(\mathsf{H})$.
- The state $\boldsymbol{h}$ cannot be a trace because there are no traces on $\mathrm{B}(\mathsf{H})$.
- It is known that in this case ($\boldsymbol{h}$ not a trace) there exists $\alpha \in \operatorname{Irr}\mathbb{G}$ with

$$(\rho_{\alpha,1}, \dots, \rho_{\alpha,n_\alpha}) \neq (1, \dots, 1).$$

- Let us assume that the set $\{\rho_{\alpha,1}, \dots, \rho_{\alpha,n_\alpha}\}$ is invariant under taking inverses.

If this doesn't hold we can construct another compact quantum group $\mathbb{H}$ out of $\mathbb{G}$ with $\alpha \in \operatorname{Irr}\mathbb{H}$ for which this holds and we still have $\mathsf{L}^\infty(\mathbb{H}) \cong \mathrm{B}(\mathsf{H})$.

Step 2.

- Let $\pi\colon \mathsf{L}^\infty(\mathbb{G}) \to \mathrm{B}(\mathsf{H})$ be the assumed isomorphism.
- The state $\boldsymbol{h}$ must be of the form

$$\boldsymbol{h}(x) = \mathrm{Tr}(Ax), \qquad x \in \mathsf{L}^\infty(\mathbb{G})$$

  for some positive trace-class operator A on H with eigenvalues $q_1 > q_2 > \cdots > 0$.
- For each n let $\mathsf{H}(A = q_n)$ be the corresponding eigenspace, so that

$$\mathsf{H} = \bigoplus_{n=1}^{\infty} \mathsf{H}(A = q_n).$$

  Moreover, we have $\dim \mathsf{H}(A = q_n) < +\infty$ for all n.
- We have

$$\pi\bigl(\sigma_t^{\boldsymbol{h}}(x)\bigr) = A^{\mathrm{i}t}\pi(x)A^{-\mathrm{i}t}, \qquad x \in \mathsf{L}^\infty(\mathbb{G}),\ t \in \mathbb{R}.$$

Step 3.

- There is a strictly positive self-adjoint operator B on H such that
$$\pi\bigl(\tau_t^{\mathbb{G}}(x)\bigr)=B^{\mathrm{i}t}\pi(x)B^{-\mathrm{i}t},\qquad x\in\mathsf{L}^{\infty}(\mathbb{G}),\ t\in\mathbb{R}$$
(this is a consequence of Stone's theorem).
- The fact that the groups $(\sigma_t^{\boldsymbol{h}})_{t\in\mathbb{R}}$ and $(\tau_t^{\mathbb{G}})_{t\in\mathbb{R}}$ commute implies that A and B strongly commute.
- Hence for any n the operator B restricts to a positive operator on the finite-dimensional Hilbert space $\mathsf{H}(A=q_n)$.
- Let $\mu_{n,1}>\cdots>\mu_{n,P_n}$ be the complete list of eigenvalues of this restriction.
- We have
$$\mathsf{H}=\bigoplus_{n=1}^{\infty}\bigoplus_{p=1}^{P_n}\mathsf{H}(A=q_n)\cap\mathsf{H}(B=\mu_{n,p}).$$

Step 4.

- Claim: $\pi(U^\alpha_{k,1})$ maps $\mathsf{H}(A=q_n)$ into $\mathsf{H}(A=\rho_{\alpha,k}\rho_{\alpha,1}q_n)$.

Indeed: take $\xi\in\mathsf{H}(A=q_n)$. Then

$$\begin{aligned}A^{\mathrm{i}t}\pi(U_{k,1})\xi&=A^{\mathrm{i}t}\pi(U_{k,1})A^{-\mathrm{i}t}A^{\mathrm{i}t}\xi=\pi\bigl(\sigma^{\boldsymbol{h}}_t(U_{k,1})\bigr)q_n^{\mathrm{i}t}\xi\\&=\pi\bigl(\rho^{\mathrm{i}t}_{\alpha,k}U_{k,1}\rho^{\mathrm{i}t}_{\alpha,1}\bigr)q_n^{\mathrm{i}t}\xi=(\rho_{\alpha,k}\rho_{\alpha,1}q_n)^{\mathrm{i}t}\pi(U_{k,1})\xi.\end{aligned}$$

- Claim: $\pi(U^\alpha_{k,1})$ maps $\mathsf{H}(B=\mu_{n,p})$ into $\mathsf{H}(B=\rho_{\alpha,k}\rho^{-1}_{\alpha,1}\mu_{n,p})$.

Indeed: take $\eta\in\mathsf{H}(B=\mu_{n,p})$. Then

$$\begin{aligned}B^{\mathrm{i}t}\pi(U_{k,1})\eta&=B^{\mathrm{i}t}\pi(U_{k,1})B^{-\mathrm{i}t}B^{\mathrm{i}t}\eta=\pi\bigl(\tau^{\mathbb{H}}_t(U_{k,1})\bigr)\mu^{\mathrm{i}t}_{n,p}\eta\\&=\pi\bigl(\rho^{\mathrm{i}t}_{\alpha,k}U_{k,1}\rho^{-\mathrm{i}t}_{\alpha,1}\bigr)\mu^{\mathrm{i}t}_{n,p}\eta=\bigl(\rho_{\alpha,k}\rho^{-1}_{\alpha,1}\mu_{n,p}\bigr)^{\mathrm{i}t}\pi(U_{k,1})\eta.\end{aligned}$$

- Let ζ be a non-zero element of $\mathsf{H}(A=q_1)\cap\mathsf{H}(B=\mu_{1,P_1})$. We will show that $\pi(U^\alpha_{k,1})\zeta=0$ for all $k\in\{1,\dots,n_\alpha\}$.

Step 4. (continued)

- By the previous claims we have

$$\pi(U^{\alpha}_{k,1})\zeta \in \mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) \cap \mathsf{H}(B = \rho_{\alpha,k}\rho_{\alpha,1}^{-1}\mu_{1,P_1}).$$

- If $\rho_{\alpha,k} = \rho_{\alpha,1}$ then $\rho_{\alpha,k}\rho_{\alpha,1}q_1 = \rho_{\alpha,1}^2 q_1 > q_1 = \|A\|$, so $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \{0\}$ and consequently $\pi(U_{k,1})\zeta = 0$.
- If $\rho_{\alpha,k} < \rho_{\alpha,1}$ then first of all

$$\rho_{\alpha,k}\rho_{\alpha,1}q_1 \geqslant \bigl(\min_i\{\rho_{\alpha,i}\}\bigr)\rho_{\alpha,1}q_1 = \rho_{\alpha,1}^{-1}\rho_{\alpha,1}q_1 = q_1$$

(invariance of $\{\rho_{\alpha,1},\dots,\rho_{\alpha,n_\alpha}\}$ under taking inverses!). Thus

$$\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \mathsf{H}(A = q_1) \quad \text{or} \quad \mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \{0\}.$$

Clearly, if $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \{0\}$ then $\pi(U^{\alpha}_{k,1})\zeta = 0$.

Step 4. (continued)

- We have $\pi(U^{\alpha}_{k,1})\zeta \in \mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) \cap \mathsf{H}(B = \rho_{\alpha,k}\rho_{\alpha,1}^{-1}\mu_{1,P_1})$ and $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \mathsf{H}(A = q_1)$ or $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \{0\}$.
- What happens if $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \mathsf{H}(A = q_1)$?

In this case $\rho_{\alpha,k}$ must be $\rho_{\alpha,1}^{-1}$, so

$$\rho_{\alpha,k}\rho_{\alpha,1}^{-1}\mu_{1,P_1} = \rho_{\alpha,1}^{-2}\mu_{1,P_1} < \mu_{1,P_1} = \min \operatorname{Sp}\bigl(B\big|_{\mathsf{H}(A=q_1)}\bigr).$$

Consequently $\mathsf{H}\bigl(B = \rho_{\alpha,k}\rho_{\alpha,1}^{-1}\mu_{1,P_1}\bigr) = \{0\}$ and

$$\pi(U_{k,1})\zeta \in \mathsf{H}(A = q_1) \cap \mathsf{H}\bigl(B = \rho_{\alpha,k}\rho_{\alpha,1}^{-1}\mu_{1,P_1}\bigr) = \{0\}.$$

In particular $\pi(U_{k,1})\zeta = 0$.

Step 5.

- We have shown that there is a non-zero $\zeta \in \mathsf{H}$ with

$$\pi(U_{k,1})\zeta, \qquad k = 1, \ldots, n_\alpha.$$

- But $U^\alpha = \begin{bmatrix} U^\alpha_{1,1} & \cdots & U^\alpha_{1,n_\alpha} \\ \vdots & \ddots & \vdots \\ U^\alpha_{n_\alpha,1} & \cdots & U^\alpha_{n_\alpha,n_\alpha} \end{bmatrix}$ is unitary matrix, so

$$0 \neq \zeta = \sum_{k=1}^{n_\alpha} \pi(U_{k,1})^* \pi(U_{k,1})\zeta = 0.$$

- This contradiction shows that the existence of $\mathbb{G}$ such that $\mathsf{L}^\infty(\mathbb{G}) \cong \mathrm{B}(\mathsf{H})$ is impossible.

□

Theorem (Jacek Krajczok & Mateusz Wasilewski)

Let $q \in]-1, 1[\setminus\{0\}$ and $\nu \in \mathbb{R}\setminus\{0\}$ and consider the action α^ν of $\mathbb{Q}$ with discrete topology on $\mathrm{SU}_q(2)$ given by

$$\alpha^\nu_r(x) = \tau^{\mathrm{SU}_q(2)}_{\nu r}(x), \qquad x \in \mathsf{L}^\infty\big(\mathrm{SU}_q(2)\big),\ r \in \mathbb{Q}.$$

Let $\mathbb{H}_{\nu,q}$ be the corresponding bicrossed product:

$$\mathbb{H}_{\nu,q} = \mathbb{Q} \bowtie \mathrm{SU}_q(2).$$

Then

1. $\mathbb{H}_{\nu,q}$ is a compact quantum group,
2. $\mathbb{H}_{\nu,q}$ is coamenable and hence $\mathsf{L}^\infty(\mathbb{H}_{\nu,q})$ is injective,
3. if $\nu \log|q| \notin \pi\mathbb{Q}$ then $\mathsf{L}^\infty(\mathbb{H}_{\nu,q})$ is the injective factor of type II_∞,
4. the spectrum of the modular operator for the Haar measure $\boldsymbol{h}_{\nu,q}$ of $\mathbb{H}_{\nu,q}$ is $\{0\} \cup q^{2\mathbb{Z}}$.

Let $\big((\nu_n,q_n)\big)_{n\in\mathbb{N}}$ be a sequence of parameters as described above ($\nu_n\log|q_n|\notin\pi\mathbb{Q}$ for all n) and consider the compact quantum group

$$\mathbb{G}=\bigtimes_{n=1}^{\infty}\mathbb{H}_{\nu_n,q_n}.$$

- In particular $\mathsf{L}^\infty(\mathbb{G})=\bigotimes_{n=1}^{\infty}\mathsf{L}^\infty(\mathbb{H}_{\nu_n,q_n})$.

EXAMPLE

If the sequence $\big((\nu_n,q_n)\big)_{n\in\mathbb{N}}$ is constant then $\mathsf{L}^\infty(\mathbb{G})$ is the injective factor of type III_{q^2} with separable predual.

- $T\big(\mathsf{L}^\infty(\mathbb{G})\big)=\frac{\pi}{\log|q|}\mathbb{Z}$,
- $S\big(\mathsf{L}^\infty(\mathbb{G})\big)=\{0\}\cup|q|^{2\mathbb{Z}}$.

EXAMPLE

If there are two subsequences $(q_{n_{1,p}})_{p\in\mathbb{N}}$ and $(q_{n_{2,p}})_{p\in\mathbb{N}}$ such that

$$\{n_{1,p}\,|\,p\in\mathbb{N}\}\cap\{n_{2,p}\,|\,p\in\mathbb{N}\}=\varnothing$$

and

$$q_{n_{1,p}}\xrightarrow[p\to\infty]{}r_1,\quad q_{n_{2,p}}\xrightarrow[p\to\infty]{}r_2$$

for some $r_1,r_2\in\,]-1,1[\,\setminus\{0\}$ such that $\frac{\pi}{\log|r_1|}\mathbb{Z}\cap\frac{\pi}{\log|r_2|}\mathbb{Z}=\{0\}$ then $\mathsf{L}^\infty(\mathbb{G})$ is the injective factor of type III_1 with separable predual.

- $T\bigl(\mathsf{L}^\infty(\mathbb{G})\bigr)=\{0\}$,
- $S\bigl(\mathsf{L}^\infty(\mathbb{G})\bigr)=\mathbb{R}_{\geqslant 0}$.

Theorem (Jacek Krajczok & P.M.S.)

There exist a family $\{\mathbb{G}_s\}_{s\in]0,1[}$ of compact quantum groups such that the von Neumann algebras $\big\{\mathsf{L}^\infty(\mathbb{G}_s)\big\}_{s\in]0,1[}$ are pairwise non-isomorphic factors of type III_0.

- $T\big(\mathsf{L}^\infty(\mathbb{G}_s)\big) \supset \mathbb{Q}$,
- defining

$$t_s = \sum_{p=1}^{\infty} \frac{\lfloor p^{1-s} \rfloor}{p!}, \qquad s \in]0,1[$$

  we have

$$\Big(\, t_{s'} \in T\big(\mathsf{L}^\infty(\mathbb{G}_s)\big) \,\Big) \iff \Big(\, s' > s \,\Big).$$

For each $\lambda \in]0,1]$ there exists uncountably many pairwise non-isomorphic compact quantum groups with $\mathsf{L}^\infty(\mathbb{G})$ the injective factor of type III_λ.

- These compact quantum groups are constructed as bicrossed products $\Gamma \bowtie \bigtimes_{n=1}^{\infty} \mathbb{H}_{\nu_n,q_n}$ with Γ a subgroup of $\mathbb{R}$ (taken with discrete topology) acting by the scaling automorphisms.
- We distinguish between them using the following invariants:
  - $T^\tau(\mathbb{G}) = \left\{t \in \mathbb{R} \,\middle|\, \tau_t^{\mathbb{G}} = \mathrm{id}\right\}$,
  - $T^\tau_{\mathrm{Inn}}(\mathbb{G}) = \left\{t \in \mathbb{R} \,\middle|\, \tau_t^{\mathbb{G}} \in \mathrm{Inn}(\mathsf{L}^\infty(\mathbb{G}))\right\}$,
  - $T^\tau_{\overline{\mathrm{Inn}}}(\mathbb{G}) = \left\{t \in \mathbb{R} \,\middle|\, \tau_t^{\mathbb{G}} \in \overline{\mathrm{Inn}}(\mathsf{L}^\infty(\mathbb{G}))\right\}$.

Thank you for your attention and please **support Ukraine!**