Integrability and quantum subgroups

Quantum groups: geometry,
representations, and beyond

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1. Quantum groups & actions
2. Homomorphisms of quantum groups
3. Quantum subgroups
4. Image & kernel
5. Integrable homomorphisms
6. Open subgroups and compact subgroups

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where $\boldsymbol{\sigma}$ is the flip.

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1. ***bicharacters*** *from* $\mathbb{H}$ *to* $\mathbb{G}$*, i.e. unitaries* $V \in L^\infty(\widehat{\mathbb{G}}) \bar{\otimes} L^\infty(\mathbb{H})$ *such that*
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A von Neumann subalgebra $\mathsf{N} \subset L^{\infty}(\mathbb{G})$ is a Baaj-Vaes subalgebra if and only if there is a locally compact quantum group $\mathbb{K}$ such that $\mathsf{N} = L^{\infty}(\mathbb{K})$ and $\Delta_{\mathbb{K}} = \Delta_{\mathbb{G}}\big|_{\mathsf{N}}$.

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- Equivalently there is a closed quantum subgroup $\widehat{\mathbb{K}}$ of $\widehat{\mathbb{G}}$ such that N is $L^\infty(\widehat{\widehat{\mathbb{K}}})$ embedded in $L^\infty(\widehat{\widehat{\mathbb{G}}}) = L^\infty(\mathbb{G})$.

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The set

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- Point 1 "means" that $\ker\Pi$ is compact.

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Theorem (K. De Commer)

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Theorem

The subgroup $\mathbb{H}$ is open if and only if β is integrable.

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THEOREM

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Corollary

1. α is integrable if and only if $\ker\Pi$ is compact and $\mathrm{im}\,\Pi$ is closed and topologically isomorphic to $H/\ker\Pi$.
2. When Π is injective, α is integrable if and only if the image of Π is closed and Π is a homeomorphism onto its image.

Thank you.