INTEGRABILITY AND QUANTUM SUBGROUPS

QUANTUM GROUPS: GEOMETRY, REPRESENTATIONS, AND BEYOND

OSLO AND AKERSHUS UNIVERSITY COLLEGE OF APPLIED SCIENCES, OSLO

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May 12, 2016

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- 2 Homomorphisms of quantum groups
- **3** QUANTUM SUBGROUPS
- 4 IMAGE & KERNEL
- 5 INTEGRABLE HOMOMORPHISMS
- 6 OPEN SUBGROUPS AND COMPACT SUBGROUPS

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- $\mathbb{G} \leftarrow --$ locally compact quantum group,
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May 12, 2016 4 / 17 An action of G on a von Neumann algebra N can be
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- Any action admits a **canonical implementation** *U* (work of S. Vaes). There is a precise criterion of integrability phrased in terms of the canonical implementation.

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THEOREM

There are bijections between sets of

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THEOREM

There are bijections between sets of

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 $(\Delta_{\widehat{\mathbb{G}}} \otimes \mathrm{id})V = V_{23}V_{13}$ and $(\mathrm{id} \otimes \Delta_{\mathbb{H}})V = V_{12}V_{13}$,

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2 **right quantum group homomorphisms** from \mathbb{H} to \mathbb{G} *i.e.* actions $\alpha: L^{\infty}(\mathbb{G}) \longrightarrow L^{\infty}(\mathbb{G}) \bar{\otimes} L^{\infty}(\mathbb{H})$ such that

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DEFINITION

A **homomorphism** $\Pi: \mathbb{H} \longrightarrow \mathbb{G}$ is an element of either of the sets (1, (2) or (3).

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${\ \bullet \ }$ Let $\Pi \colon \mathbb{H} \to \mathbb{G}$ be a homomorphism

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 \bullet Let $\Pi \colon \mathbb{H} \to \mathbb{G}$ be a homomorphism with corresponding bicharacter

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and right quantum group homomorphism

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$$\alpha\colon L^{\infty}(\mathbb{G}) \longrightarrow L^{\infty}(\mathbb{G}) \bar{\otimes} \ L^{\infty}(\mathbb{H}).$$

Then α is implemented by *V*:

$$\alpha(\mathbf{x}) = V(\mathbf{x} \otimes \mathbb{1})V^*, \qquad \mathbf{x} \in L^{\infty}(\mathbb{G}).$$

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$$\alpha \in \mathrm{Mor}\big(\mathrm{C}_{0}(\mathbb{G}), \mathrm{C}_{0}(\mathbb{G}) \otimes \mathrm{C}_{0}(\mathbb{H})\big).$$

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A von Neumann subalgebra $N \subset L^{\infty}(\mathbb{G})$ is a Baaj-Vaes subalgebra if and only if there is a locally compact quantum group \mathbb{K} such that $N = L^{\infty}(\mathbb{K})$ and $\Delta_{\mathbb{K}} = \Delta_{\mathbb{G}}|_{N}$.

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 ■ Equivalently there is a closed quantum subgroup K of G
 such that N is L[∞](Â) embedded in L[∞](Â) = L[∞](G).

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IMAGE

• $\Pi \colon \mathbb{H} \longrightarrow \mathbb{G} \leftarrow --$ homomorphism

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• $\Pi: \mathbb{H} \longrightarrow \mathbb{G} \leftarrow -$ homomorphism, $V \leftarrow -$ corresponding bicharacter.

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PROPOSITION

The set

$$\mathsf{L} = \left\{ (\mathrm{id} \otimes \zeta) V \middle| \zeta \in \mathrm{B}(L^2(\mathbb{H}))_* \right\}^{\sigma}$$
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PROPOSITION

The set

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PROPOSITION

The set

$$\mathsf{R} = \left\{ (\phi \otimes \mathrm{id}) V \middle| \phi \in \mathrm{B}(L^2(\mathbb{G}))_*
ight\}^{\sigma}$$

is a Baaj-Vaes subalgebra of $L^{\infty}(\mathbb{H})$.

DEFINITION

The quantum group $\mathbb{H}/ker\,\Pi$ is the quantum group related to the Baaj-Vaes subalgebra R. In particular

$$L^{\infty}(\mathbb{H}/\ker\Pi) = \{(\phi \otimes \mathrm{id})V | \phi \in \mathrm{B}(L^{2}(\mathbb{G}))_{*}\}^{\sigma}.$$

 One can also define H/ker α for an action α of H on a von Neumann algebra. If α corresponds to Π we have H/ker Π = H/ker α. If α is free, we have H/ker α = H.

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• $\Pi : \mathbb{H} \longrightarrow \mathbb{G}$, *V*, $\alpha \leftarrow --$ as before.

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- $\Pi : \mathbb{H} \longrightarrow \mathbb{G}$, *V*, $\alpha \leftarrow --$ as before.
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• Note:

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- Note:
 - $V \in L^{\infty}(\widehat{\operatorname{im}\Pi}) \bar{\otimes} L^{\infty}(\mathbb{H}/\ker \Pi),$

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• $V \in L^{\infty}(\widehat{\operatorname{im}\Pi}) \bar{\otimes} L^{\infty}(\mathbb{H}/\ker \Pi),$

• an isomorphism χ as above is necessarily unique.

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THEOREM

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INTEGRABILITY AND QUANTUM SUBGROUPS

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• Point 1 "means" that ker Π is compact.

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(ロト・イラト・イミト・ミークへへ) P.M. SOŁTAN (WARSAW) INTEGRABILITY AND QUANTUM SUBGROUPS MAY 12, 2016 13 / 17

Let $\Pi \colon \mathbb{H} \longrightarrow \mathbb{G}$ identify \mathbb{H} with a closed quantum subgroup of \mathbb{G} .

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Let $\Pi \colon \mathbb{H} \longrightarrow \mathbb{G}$ identify \mathbb{H} with a closed quantum subgroup of \mathbb{G} . Then the associated action α is integrable.

COROLLARY

A Woronowicz-closed quantum subgroup $\mathbb H$ of $\mathbb G$ is a closed quantum subgroup if and only if the corresponding action

 $\alpha\colon L^{\infty}(\mathbb{G}) \longrightarrow L^{\infty}(\mathbb{G}) \bar{\otimes} L^{\infty}(\mathbb{H})$

is integrable.

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PROOF.

- \Rightarrow Immediate from theorem.
- \Leftarrow : Woronowicz-closedness implies that α is free

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PROOF.

 \Rightarrow Immediate from theorem.

 \Leftarrow : Woronowicz-closedness implies that *α* is free, so $\mathbb{H}/\ker \Pi = \mathbb{H}$.

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PROOF.

- \Rightarrow Immediate from theorem.
- \Leftarrow : Woronowicz-closedness implies that *α* is free, so $\mathbb{H}/\ker \Pi = \mathbb{H}$. Integrability implies that $\mathbb{H}/\ker \Pi \cong \overline{\operatorname{im} \Pi}$

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- \Rightarrow Immediate from theorem.
- \Leftarrow : Woronowicz-closedness implies that α is free, so

 $\mathbb{H}/\ker \Pi = \mathbb{H}$. Integrability implies that $\mathbb{H}/\ker \Pi \cong \overline{\operatorname{im} \Pi}$ and $\overline{\operatorname{im} \Pi}$ is closed.

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(ロト・イラト・イミト・マシト・ション・クへへ) P.M. Soltan (Warsaw) Integrability and guantum subgroups May 12, 2016 14 / 17

• Let \mathbb{H} be a closed subgroup of \mathbb{G} .

(Warsaw) INTEGRABILITY AND QUANTUM SUBGROUPS MAY 12, 2016 14 / 17

- Let \mathbb{H} be a closed subgroup of \mathbb{G} .
- $L^{\infty}(\widehat{\mathbb{H}})$ is an invariant subalgebra of $L^{\infty}(\widehat{\mathbb{G}})$

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THEOREM

The subgroup $\mathbb H$ is open if and only if β is integrable.

P.M. SOŁTAN (WARSAW)

INTEGRABILITY AND QUANTUM SUBGROUPS

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P.M. SOŁTAN (WARSAW)

May 12, 2016 15 / 17

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MAY 12, 2016 16 / 17 INTEGRABILITY AND QUANTUM SUBGROUPS

P.M. SOŁTAN (WARSAW)

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COROLLARY

1 α is integrable if and only if ker Π is compact and im Π is closed and topologically isomorphic to *H*/ker Π .

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COROLLARY

- **1** α is integrable if and only if ker Π is compact and im Π is closed and topologically isomorphic to *H*/ker Π .
- **2** When Π is injective, α is integrable if and only if the image of Π is closed and Π is a homeomorphism onto its image.

P.M. SOŁTAN (WARSAW)

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Thank you.

P.M. SOLTAN (WARSAW)

INTEGRABILITY AND QUANTUM SUBGROUPS

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