Why $\mathrm{B}(\ell^2)$ is not $L^\infty(\mathbb{G})$

Quantum Groups: Current Trends and New Perspectives

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1. Compact quantum groups

2. The structure of $L^\infty(\mathbb{G})$

3. What if $L^\infty(\mathbb{G}) \cong \mathrm{B}(\ell^2)$?

4. Other injective factors

The basics

Definition

A **compact quantum group** $\mathbb{G}$ is described by

- a von Neumann algebra $L^\infty(\mathbb{G})$,
- a unital $*$-homomorphism $\Delta\colon L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{G})$

(continuous in the σ-weak topology) such that

- $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$,
- there exists a faithful state $\boldsymbol{h}$ on $L^\infty(\mathbb{G})$ such that

$$\forall\, x \in L^\infty(\mathbb{G}) \quad (\boldsymbol{h} \otimes \mathrm{id})\Delta(x) = \boldsymbol{h}(x)\mathbb{1} = (\mathrm{id} \otimes \boldsymbol{h})\Delta(x). \qquad (\heartsuit)$$

The condition $(\heartsuit)$ determines $\boldsymbol{h}$ uniquely. We call this state the **Haar measure** of $\mathbb{G}$.

DEFINITION

Let $\mathbb{G}$ be a compact quantum group. A **finite-dimensional unitary representation** of $\mathbb{G}$ is a unitary $U \in \mathrm{B}(\mathsf{H}) \otimes L^\infty(\mathbb{G})$ (with H a finite-dimensional Hilbert space) such that

$$(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13},$$

where

- $U_{12} = U \otimes \mathbb{1} \in \mathrm{B}(\mathsf{H}) \otimes L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})$,
- $U_{13} = (\mathrm{id} \otimes \mathrm{flip})(U_{12}) \in \mathrm{B}(\mathsf{H}) \otimes L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})$.

- We say that a representation $U \in \mathrm{B}(\mathsf{H}) \otimes L^\infty(\mathbb{G})$ is **irreducible** if $(T \otimes \mathbb{1})U = U(T \otimes \mathbb{1})$ implies $T = \lambda \mathbb{1}_{\mathsf{H}}$.
- Representations $U \in \mathrm{B}(\mathsf{H}) \otimes L^\infty(\mathbb{G})$ and $V \in \mathrm{B}(\mathsf{K}) \otimes L^\infty(\mathbb{G})$ are **equivalent** if there is a unitary $S \in \mathrm{B}(\mathsf{H}, \mathsf{K})$ such that $(S \otimes \mathbb{1})U = V(S \otimes \mathbb{1})$.

MATRIX ELEMENTS OF IRREPS

- Let $U \in \mathrm{B}(\mathsf{H}) \otimes L^\infty(\mathbb{G})$ be a representation. Then any $\omega \in \mathrm{B}(\mathsf{H})^*$ defines $(\omega \otimes \mathrm{id})(U) \in L^\infty(\mathbb{G})$ which is called a **matrix element** or a **coefficient** of U.
- Typically we take $\omega(\cdot) = \langle \xi | \cdot | \eta \rangle$ for some vectors $\xi, \eta \in \mathsf{H}$.
- Choosing an orthonormal basis $\{\xi_1, \ldots, \xi_n\}$ of H yields $U_{i,j} = (\omega_{i,j} \otimes \mathrm{id})(U)$ where $\omega_{i,j} = \langle \xi_i | \cdot | \xi_j \rangle$.

From now on we denote by $\mathrm{Irr}(\mathbb{G})$ the set of equivalence classes of irreps of $\mathbb{G}$. For each $\alpha \in \mathrm{Irr}(\mathbb{G})$ we fix $U^\alpha \in \alpha$. Then any orthonormal basis $\xi_1^\alpha, \ldots, \xi_{n_\alpha}^\alpha$ of the carrier Hilbert space H^α of U^α defines the matrix elements $U_{i,j}^\alpha$.

THEOREM

$\mathrm{span}\left\{ U_{i,j}^\alpha \,\middle|\, \alpha \in \mathrm{Irr}(\mathbb{G}),\ i,j \in \{1, \ldots, n_\alpha\} \right\}$ *is σ-weakly dense in* $L^\infty(\mathbb{G})$.

The ρ-operators

- For each $\alpha\in\mathrm{Irr}(\mathbb{G})$ let $V^{\alpha}=(j\otimes\mathrm{id})\bigl(U^{\alpha*}\bigr)\in\mathrm{B}(\mathsf{H}^{\alpha*})\otimes L^{\infty}(\mathbb{G})$ ($j\colon\ \mathrm{B}(\mathsf{H}^{\alpha})\to\mathrm{B}(\mathsf{H}^{\alpha*})$ maps T to the operator $\langle\psi|\mapsto\langle T^{*}\psi|$).
- Next we let $\rho_{\alpha}=\mathrm{const}\cdot j\bigl((\mathrm{id}\otimes\boldsymbol{h})\bigl(V^{\alpha*}V^{\alpha}\bigr)\bigr)$ with the constant chosen so that $\mathrm{Tr}(\rho_{\alpha})=\mathrm{Tr}(\rho_{\alpha}^{-1})$.
- Note that ρ_{α} is positive.

From now on for each $\alpha\in\mathrm{Irr}(\mathbb{G})$ we fix an orthonormal basis of H^{α} in which ρ_{α} is diagonal:

$$\rho_{\alpha}=\begin{bmatrix}\rho_{\alpha,1}&&\\&\ddots&\\&&\rho_{\alpha,n_{\alpha}}\end{bmatrix}$$

and $\rho_{\alpha,1}\geqslant\cdots\geqslant\rho_{\alpha,n_{\alpha}}$.

- We have $\boldsymbol{h}\bigl(U^{\alpha}_{k,l}{}^{*}U^{\beta}_{i,j}\bigr)=\delta_{\alpha\beta}\frac{\delta_{ki}\rho^{-1}_{\alpha,j}\delta_{l,j}}{\mathrm{Tr}(\rho_{\alpha})}$, so $\bigl\{U^{\alpha}_{i,j}\bigr\}$ are linearly independent.

The modular group and the scaling group

Theorem

There exist two σ-weakly continuous one-parameter groups $\sigma^{\boldsymbol{h}}$ and $\tau^{\mathbb{G}}$ of automorphisms of $L^\infty(\mathbb{G})$ such that

$$\tau_t^{\mathbb{G}}(U_{i,j}^{\alpha}) = \rho_{\alpha,i}^{\mathrm{i}t} U_{i,j}^{\alpha} \rho_{\alpha,j}^{-\mathrm{i}t}$$

$$\sigma_t^{\boldsymbol{h}}(U_{i,j}^{\alpha}) = \rho_{\alpha,i}^{\mathrm{i}t} U_{i,j}^{\alpha} \rho_{\alpha,j}^{\mathrm{i}t}$$

for all $\alpha \in \mathrm{Irr}(\mathbb{G})$, $i,j \in \{1,\dots,n_\alpha\}$ and $t \in \mathbb{R}$.

- Clearly the two groups commute.

What if?

Suppose that there is a compact quantum group $\mathbb{G}$ such that $L^\infty(\mathbb{G}) \cong \mathrm{B}(\mathsf{H})$, where H is a Hilbert space such that $\dim \mathsf{H} > 1$.

- If H were finite-dimensional then $\mathrm{B}(\mathsf{H})$ would be simple, but a finite dimensional $L^\infty(\mathbb{G})$ admits a character, so this is impossible.
- The case $\dim \mathsf{H} > \aleph_0$ is ruled out by the fact that there are no faithful normal states on $\mathrm{B}(\mathsf{H})$ for non-separable H.
- Thus we are left with $\mathsf{H} \cong \ell^2$.
- We will show that this leads to a contradiction.

Step 1.

- Suppose $\mathbb{G}$ is a compact quantum group with $L^\infty(\mathbb{G}) \cong \mathrm{B}(\mathsf{H})$.
- The state ***h*** cannot be a trace because there are no traces on $\mathrm{B}(\mathsf{H})$.
- It is known that in this case (***h*** not a trace) there exists $\alpha \in \mathrm{Irr}(\mathbb{G})$ with

$$(\rho_{\alpha,1}, \ldots, \rho_{\alpha,n_\alpha}) \neq (1, \ldots, 1).$$

- Let us assume that the set $\{\rho_{\alpha,1}, \ldots, \rho_{\alpha,n_\alpha}\}$ is invariant under taking inverses.

If this doesn't hold we can show that the compact quantum group $\mathbb{G} \times \mathbb{G}$ has $\beta \in \mathrm{Irr}(\mathbb{G} \times \mathbb{G})$ such that ρ_β is non-trivial and $\{\rho_{\beta,1}, \ldots, \rho_{\beta,n_\beta}\} = \{\rho_{\beta,1}^{-1}, \ldots, \rho_{\beta,n_\beta}^{-1}\}$.
Still $L^\infty(\mathbb{G} \times \mathbb{G}) = L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} L^\infty(\mathbb{G}) \cong \mathrm{B}(\mathsf{H}) \mathbin{\overline{\otimes}} \mathrm{B}(\mathsf{H}) \cong \mathrm{B}(\mathsf{H})$.

Step 2.

- Let $\pi\colon L^\infty(\mathbb{G}) \to B(\mathsf{H})$ be the assumed isomorphism.
- The state $\boldsymbol{h}$ must be of the form

$$\boldsymbol{h}(x) = \operatorname{Tr}\bigl(A\pi(x)\bigr), \qquad x \in L^\infty(\mathbb{G})$$

  for some positive trace-class operator A on H with eigenvalues $q_1 > q_2 > \cdots > 0$.
- For each n let $\mathsf{H}(A = q_n)$ be the corresponding eigenspace, so that

$$\mathsf{H} = \bigoplus_{n=1}^{\infty} \mathsf{H}(A = q_n).$$

  Moreover, we have $\dim \mathsf{H}(A = q_n) < +\infty$ for all n.
- We have

$$\pi\bigl(\sigma_t^{\boldsymbol{h}}(x)\bigr) = A^{it}\pi(x)A^{-it}, \qquad x \in L^\infty(\mathbb{G}),\ t \in \mathbb{R}.$$

Step 3.

- There is a strictly positive self-adjoint operator B on H such that
$$\pi\bigl(\tau_t^{\mathbb{G}}(x)\bigr) = B^{\mathrm{i}t}\pi(x)B^{-\mathrm{i}t}, \qquad x \in L^\infty(\mathbb{G}),\ t \in \mathbb{R}$$
(this is a consequence of Stone's theorem).
- The fact that the groups $(\sigma_t^{\boldsymbol{h}})_{t\in\mathbb{R}}$ and $(\tau_t^{\mathbb{G}})_{t\in\mathbb{R}}$ commute implies that A and B strongly commute.
- Hence for any n the operator B restricts to a positive operator on the finite-dimensional Hilbert space $\mathsf{H}(A = q_n)$.
- Let $\mu_{n,1} > \cdots > \mu_{n,P_n}$ be the complete list of eigenvalues of this restriction.
- We have
$$\mathsf{H} = \bigoplus_{n=1}^{\infty} \bigoplus_{p=1}^{P_n} \mathsf{H}(A = q_n) \cap \mathsf{H}(B = \mu_{n,p}).$$

Step 4.

- Claim: $\pi(U^\alpha_{k,1})$ maps $\mathsf{H}(A=q_n)$ into $\mathsf{H}(A=\rho_{\alpha,k}\rho_{\alpha,1}q_n)$.

Indeed: take $\xi\in\mathsf{H}(A=q_n)$. Then

$$\begin{aligned}A^{\mathrm{it}}\pi(U^\alpha_{k,1})\xi &= A^{\mathrm{it}}\pi(U^\alpha_{k,1})A^{-\mathrm{it}}A^{\mathrm{it}}\xi = \pi\bigl(\sigma^{\boldsymbol{h}}_t(U^\alpha_{k,1})\bigr)q_n^{\mathrm{it}}\xi\\ &= \pi\bigl(\rho^{\mathrm{it}}_{\alpha,k}U^\alpha_{k,1}\rho^{\mathrm{it}}_{\alpha,1}\bigr)q_n^{\mathrm{it}}\xi = (\rho_{\alpha,k}\rho_{\alpha,1}q_n)^{\mathrm{it}}\pi(U^\alpha_{k,1})\xi.\end{aligned}$$

- Claim: $\pi(U^\alpha_{k,1})$ maps $\mathsf{H}(B=\mu_{n,p})$ into $\mathsf{H}(B=\rho_{\alpha,k}\rho^{-1}_{\alpha,1}\mu_{n,p})$.

Indeed: take $\eta\in\mathsf{H}(B=\mu_{n,p})$. Then

$$\begin{aligned}B^{\mathrm{it}}\pi(U^\alpha_{k,1})\eta &= B^{\mathrm{it}}\pi(U^\alpha_{k,1})B^{-\mathrm{it}}B^{\mathrm{it}}\eta = \pi\bigl(\tau^{\mathbb{H}}_t(U^\alpha_{k,1})\bigr)\mu^{\mathrm{it}}_{n,p}\eta\\ &= \pi\bigl(\rho^{\mathrm{it}}_{\alpha,k}U^\alpha_{k,1}\rho^{-\mathrm{it}}_{\alpha,1}\bigr)\mu^{\mathrm{it}}_{n,p}\eta = \bigl(\rho_{\alpha,k}\rho^{-1}_{\alpha,1}\mu_{n,p}\bigr)^{\mathrm{it}}\pi(U^\alpha_{k,1})\eta.\end{aligned}$$

- Let ζ be a non-zero element of $\mathsf{H}(A=q_1)\cap\mathsf{H}(B=\mu_{1,P_1})$. We will show that $\pi(U^\alpha_{k,1})\zeta=0$ for all $k\in\{1,\dots,n_\alpha\}$.

Step 4. (continued)

- By the previous claims we have

$$\pi(U^\alpha_{k,1})\zeta \in \mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) \cap \mathsf{H}(B = \rho_{\alpha,k}\rho^{-1}_{\alpha,1}\mu_{1,P_1}).$$

- If $\rho_{\alpha,k} = \rho_{\alpha,1}$ then $\rho_{\alpha,k}\rho_{\alpha,1}q_1 = \rho^2_{\alpha,1}q_1 > q_1 = \|A\|$, so $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \{0\}$ and consequently $\pi(U^\alpha_{k,1})\zeta = 0$.
- If $\rho_{\alpha,k} < \rho_{\alpha,1}$ then first of all

$$\rho_{\alpha,k}\rho_{\alpha,1}q_1 \geqslant \big(\min_i\{\rho_{\alpha,i}\}\big)\rho_{\alpha,1}q_1 = \rho^{-1}_{\alpha,1}\rho_{\alpha,1}q_1 = q_1$$

(invariance of $\{\rho_{\alpha,1},\dots,\rho_{\alpha,n_\alpha}\}$ under taking inverses!). Thus

$$\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \mathsf{H}(A = q_1) \quad \text{or} \quad \mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \{0\}.$$

Clearly, if $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \{0\}$ then $\pi(U^\alpha_{k,1})\zeta = 0$.

Step 4. (continued further)

- We have $\pi(U^\alpha_{k,1})\zeta \in \mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) \cap \mathsf{H}(B = \rho_{\alpha,k}\rho_{\alpha,1}^{-1}\mu_{1,P_1})$ and $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \mathsf{H}(A = q_1)$ or $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \{0\}$.
- What happens if $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \mathsf{H}(A = q_1)$?

In this case $\rho_{\alpha,k}$ must be $\rho_{\alpha,1}^{-1}$, so

$$\rho_{\alpha,k}\rho_{\alpha,1}^{-1}\mu_{1,P_1} = \rho_{\alpha,1}^{-2}\mu_{1,P_1} < \mu_{1,P_1} = \min \mathrm{Sp}\big(B\big|_{\mathsf{H}(A=q_1)}\big).$$

Consequently $\mathsf{H}\big(B = \rho_{\alpha,k}\rho_{\alpha,1}^{-1}\mu_{1,P_1}\big) = \{0\}$ and

$$\pi(U^\alpha_{k,1})\zeta \in \mathsf{H}(A = q_1) \cap \mathsf{H}\big(B = \rho_{\alpha,k}\rho_{\alpha,1}^{-1}\mu_{1,P_1}\big) = \{0\}.$$

In particular $\pi(U^\alpha_{k,1})\zeta = 0$.

Step 5.

- We have shown that there is a non-zero $\zeta \in \mathsf{H}$ with

$$\pi(U^\alpha_{k,1})\zeta, \qquad k = 1, \ldots, n_\alpha.$$

- But $U^\alpha = \begin{bmatrix} U^\alpha_{1,1} & \cdots & U^\alpha_{1,n_\alpha} \\ \vdots & \ddots & \vdots \\ U^\alpha_{n_\alpha,1} & \cdots & U^\alpha_{n_\alpha,n_\alpha} \end{bmatrix}$ is unitary matrix, so

$$0 \neq \zeta = \sum_{k=1}^{n_\alpha} \pi(U^\alpha_{k,1})^* \pi(U^\alpha_{k,1})\zeta = 0.$$

- This contradiction shows that the existence of $\mathbb{G}$ such that $L^\infty(\mathbb{G}) \cong B(\mathsf{H})$ is impossible.

$\square$

REMARKS

1. The proof can be tweaked to obtain

THEOREM (J. KRAJCZOK & P.M.S.)

There does not exist a compact quantum group $\mathbb{G}$ *such that* $L^\infty(\mathbb{G}) \cong \mathsf{N} \oplus B(\mathsf{H})$ *with* N *an arbitrary von Neumann algebra or the zero vector space and* H *of infinite dimension.*

REMARKS

2. Similar techniques yield the following

THEOREM (A. CHIRVASITU, J. KRAJCZOK & P.M.S.)

Let $\mathbb{G}$ be a compact quantum group such that the C^*-algebra $C(\mathbb{G})$ fits into the exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{N} \mathcal{K}(\mathsf{H}_i) \longrightarrow C(\mathbb{G}) \longrightarrow C(X) \longrightarrow 0$$

with X a compact space. The $\mathbb{G}$ is finite $(\dim C(\mathbb{G}) < +\infty)$.

3. It follows that the Podleś spheres and the quantum disk do not admit a structure of a compact quantum group.

THEOREM (J. KRAJCZOK & M. WASILEWSKI)

Let $q \in]-1,1[\setminus\{0\}$ and $\nu \in \mathbb{R}\setminus\{0\}$ and consider the action α^ν of $\mathbb{Q}$ with discrete topology on $\mathrm{SU}_q(2)$ given by

$$\alpha^\nu_r(x) = \tau^{\mathrm{SU}_q(2)}_{\nu r}(x), \qquad x \in L^\infty\bigl(\mathrm{SU}_q(2)\bigr),\ r \in \mathbb{Q}.$$

Let $\mathbb{H}_{\nu,q}$ be the corresponding bicrossed product:

$$\mathbb{H}_{\nu,q} = \mathbb{Q} \bowtie \mathrm{SU}_q(2).$$

Then

1. $\mathbb{H}_{\nu,q}$ is a compact quantum group,
2. $\mathbb{H}_{\nu,q}$ is coamenable and hence $L^\infty(\mathbb{H}_{\nu,q})$ is injective,
3. if $\nu \log|q| \notin \pi\mathbb{Q}$ then $L^\infty(\mathbb{H}_{\nu,q})$ is the injective factor of type II_∞,
4. the spectrum of the modular operator for the Haar measure $\boldsymbol{h}_{\nu,q}$ of $\mathbb{H}_{\nu,q}$ is $\{0\} \cup q^{2\mathbb{Z}}$.

Let $\big((\nu_n, q_n)\big)_{n\in\mathbb{N}}$ be a sequence of parameters as described above ($\nu_n \log|q_n| \notin \pi\mathbb{Q}$ for all n) and consider the compact quantum group

$$\mathbb{G} = \bigtimes_{n=1}^{\infty} \mathbb{H}_{\nu_n,q_n}.$$

- In particular $L^\infty(\mathbb{G}) = \bigotimes_{n=1}^{\infty} L^\infty(\mathbb{H}_{\nu_n,q_n})$.

Example

If the sequence $\big((\nu_n, q_n)\big)_{n\in\mathbb{N}}$ is constant then $L^\infty(\mathbb{G})$ is the injective factor of type III_{q^2} with separable predual.

- $T\big(L^\infty(\mathbb{G})\big) = \frac{\pi}{\log|q|}\mathbb{Z}$,
- $S\big(L^\infty(\mathbb{G})\big) = \{0\} \cup |q|^{2\mathbb{Z}}$.

Example

If there are two subsequences $(q_{n_{1,p}})_{p\in\mathbb{N}}$ and $(q_{n_{2,p}})_{p\in\mathbb{N}}$ such that

$$\{n_{1,p}\,|\,p\in\mathbb{N}\}\cap\{n_{2,p}\,|\,p\in\mathbb{N}\}=\varnothing$$

and

$$q_{n_{1,p}}\xrightarrow[p\to\infty]{}r_1,\quad q_{n_{2,p}}\xrightarrow[p\to\infty]{}r_2$$

for some $r_1,r_2\in\,]-1,1[\,\setminus\{0\}$ such that $\frac{\pi}{\log|r_1|}\mathbb{Z}\cap\frac{\pi}{\log|r_2|}\mathbb{Z}=\{0\}$ then $L^\infty(\mathbb{G})$ is the injective factor of type III_1 with separable predual.

- $T\big(L^\infty(\mathbb{G})\big)=\{0\}$,
- $S\big(L^\infty(\mathbb{G})\big)=\mathbb{R}_{\geqslant 0}$.

Theorem (J. Krajczok & P.M.S.)

There exist a family $\{\mathbb{G}_s\}_{s\in]0,1[}$ of compact quantum groups such that the von Neumann algebras $\bigl\{L^\infty(\mathbb{G}_s)\bigr\}_{s\in]0,1[}$ are pairwise non-isomorphic factors of type III_0.

- $T\bigl(L^\infty(\mathbb{G}_s)\bigr)\supset\mathbb{Q}$,
- defining

$$t_s=\sum_{p=1}^{\infty}\frac{\lfloor p^{1-s}\rfloor}{p!},\qquad s\in\,]0,1[$$

  we have

$$\Bigl(\,t_{s'}\in T\bigl(L^\infty(\mathbb{G}_s)\bigr)\,\Bigr)\iff\Bigl(\,s'>s\,\Bigr).$$

For each $\lambda \in]0,1]$ there exists uncountably many pairwise non-isomorphic compact quantum groups with $L^\infty(\mathbb{G})$ the injective factor of type III_λ.

- These compact quantum groups are constructed as bicrossed products $\Gamma \bowtie \bigtimes_{n=1}^{\infty} \mathbb{H}_{\nu_n,q_n}$ with Γ a subgroup of $\mathbb{R}$ (taken with discrete topology) acting by the scaling automorphisms.
- We distinguish between them using the following invariants:
  - $T^\tau(\mathbb{G}) = \left\{t \in \mathbb{R} \,\middle|\, \tau_t^{\mathbb{G}} = \mathrm{id}\right\}$,
  - $T^\tau_{\mathrm{Inn}}(\mathbb{G}) = \left\{t \in \mathbb{R} \,\middle|\, \tau_t^{\mathbb{G}} \in \mathrm{Inn}(L^\infty(\mathbb{G}))\right\}$,
  - $T^\tau_{\overline{\mathrm{Inn}}}(\mathbb{G}) = \left\{t \in \mathbb{R} \,\middle|\, \tau_t^{\mathbb{G}} \in \overline{\mathrm{Inn}}(L^\infty(\mathbb{G}))\right\}$.

Thank you for your attention