Exotic quantum group norms from property (T)

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Compact quantum groups

Definition

$$\mathbb{G} = \big(\mathrm{C}(\mathbb{G}), \Delta\big)$$

- $\mathrm{C}(\mathbb{G})$ — unital C*-algebra
- $\Delta\colon\ \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$

$$\begin{array}{ccc} \mathrm{C}(\mathbb{G}) & \xrightarrow{\ \Delta\ } & \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G}) \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G}) & \xrightarrow[\mathrm{id}\otimes\Delta]{} & \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G}) \end{array}$$

- $\Delta\big(\mathrm{C}(\mathbb{G})\big)\big(\mathbf{1} \otimes \mathrm{C}(\mathbb{G})\big) = \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$
- $\big(\mathrm{C}(\mathbb{G}) \otimes \mathbf{1}\big)\Delta\big(\mathrm{C}(\mathbb{G})\big) = \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$

Examples

- G — compact group,
  - $\mathrm{C}(\mathbb{G}) := \mathrm{C}(G)$
  - $\Delta(f)(x, y) = f(xy)$
- Γ – discrete group
  - $\mathrm{C}(\mathbb{G}) := \mathrm{C}^*(\Gamma)$
  - $\Delta(\gamma) = \gamma \otimes \gamma$

  or
  - $\mathrm{C}(\mathbb{G}) := \mathrm{C}^*_{\mathrm{r}}(\Gamma)$
  - $\Delta(\gamma) = \gamma \otimes \gamma$

Representations & Hopf algebra

Let $\mathbb{G}$ be a compact quantum group. A **representation** of $\mathbb{G}$ is a unitary matrix

$$u = \begin{bmatrix} u_{1,1} & \cdots & u_{1,n} \\ \vdots & \ddots & \vdots \\ u_{n,1} & \cdots & u_{n,n} \end{bmatrix} \in M_n\big(\mathrm{C}(\mathbb{G})\big)$$

such that $\Delta(u_{i,j}) = \sum_{k=1}^{n} u_{i,k} \otimes u_{k,j}$.

- The elements $\{u_{i,j}\}$ are the **matrix elements** of u.
- u is **irreducible** if it does not commute with any nontrivial (scalar) projection.
- The span $\mathrm{Pol}(\mathbb{G})$ of matrix elements of all irreducible representations of $\mathbb{G}$ is a Hopf algebra dense in $\mathrm{C}(\mathbb{G})$.

From compact to discrete

- $\mathrm{Irr}(\mathbb{G})$ — set of equivalence classes of irreps of $\mathbb{G}$.
- Chose unitary representative u^α for each $\alpha \in \mathrm{Irr}(\mathbb{G})$.
- Then $u^\alpha \in M_{n_\alpha}\big(\mathrm{Pol}(\mathbb{G})\big) \subset M_{n_\alpha}\big(\mathrm{C}(\mathbb{G})\big)$.
- Define

$$\mathrm{c}_0(\widehat{\mathbb{G}}) = \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} M_{n_\alpha}(\mathbb{C})$$

  and

$$\boldsymbol{w} = \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} u^\alpha \in \mathrm{M}\big(\,\mathrm{c}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}(\mathbb{G})\big).$$

- There exists a unique comultiplication $\widehat{\Delta}$ on $\mathrm{c}_0(\widehat{\mathbb{G}})$ such that

$$(\widehat{\Delta} \otimes \mathrm{id})\boldsymbol{w} = \boldsymbol{w}_{23}\boldsymbol{w}_{13}.$$

- $\widehat{\mathbb{G}} = \big(\mathrm{c}_0(\widehat{\mathbb{G}}), \widehat{\Delta}\big)$ is a l.c.q.g. called the **dual** of $\mathbb{G}$.
- $\widehat{\mathbb{G}}$ is a **discrete quantum group**.

OTHER COMPLETIONS OF $\mathrm{Pol}(\mathbb{G})$

- maximal (universal) C*-norm
  $\rightsquigarrow$ the completion: $\mathrm{C}(\mathbb{G}_{\mathrm{max}})$
- minimal (reduced) C*-norm
  $\rightsquigarrow$ the completion: $\mathrm{C}(\mathbb{G}_{\mathrm{min}})$
- $\|a\|_{\sim} = \max\{\|a\|, |\epsilon(a)|\}$
  $\rightsquigarrow$ the completion: $\mathrm{C}(\widetilde{\mathbb{G}})$

Example: $\mathrm{Pol}(\mathbb{G}) = \mathbb{C}[\Gamma]$

$\rightsquigarrow$ $\mathrm{C}(\mathbb{G}_{\mathrm{max}}) = \mathrm{C}^*_{\mathrm{full}}(\Gamma)$

$\rightsquigarrow$ $\mathrm{C}(\mathbb{G}_{\mathrm{min}}) = \mathrm{C}^*_{\mathrm{r}}(\Gamma)$

$\rightsquigarrow$ $\mathrm{C}(\widetilde{\mathbb{G}}) = ??$

DEFINITION

A C*-*norm on* $\mathrm{Pol}(\mathbb{G})$ *is a* ***quantum group norm*** *if*

$$\Delta\colon \mathrm{Pol}(\mathbb{G}) \longrightarrow \mathrm{Pol}(\mathbb{G}) \otimes \mathrm{Pol}(\mathbb{G})$$

extends to completions.

FACT

All of the above C*-*norms are quantum group norms.*

Exotic completions

- We are interested in quantum group norms **quantum group norms** on $\mathrm{Pol}(\mathbb{G})$ such that if $\mathrm{C}(\mathbb{G})$ is the completion we have
  - $\mathrm{C}(\mathbb{G}_{\mathrm{min}}) \neq \mathrm{C}(\mathbb{G})$,
  - $\mathrm{C}(\mathbb{G}) \neq \mathrm{C}(\mathbb{G}_{\mathrm{max}})$,
  - $\mathrm{C}(\mathbb{G}) \neq \mathrm{C}(\widetilde{\mathbb{G}}) \neq \mathrm{C}(\mathbb{G}_{\mathrm{max}})$

  (in the sense that the canonical epimorphisms are not isomorphisms).
- Another interesting possibility is
  - $\mathrm{C}(\mathbb{G}) \neq \mathrm{C}(\widetilde{\mathbb{G}}) = \mathrm{C}(\mathbb{G}_{\mathrm{max}})$.

- We call such norms **exotic** quantum group norms.
- Their existence of exotic norms is interesting for the theory of quantum group actions.

Corepresentations of $\widehat{\mathbb{G}}$

Definition

*A **corepresentation** of $\widehat{\mathbb{G}}$ is a unitary V of the form*

$$V = (\mathrm{id} \otimes \pi)\boldsymbol{w} \in \mathrm{M}\big(\mathrm{c}_0(\widehat{\mathbb{G}}) \otimes \mathscr{K}(\mathscr{H})\big),$$

where π is a representation of $\mathrm{C}(\mathbb{G}_{\max})$ on the Hilbert space $\mathscr{H}$.

- Recall: $\boldsymbol{w} = \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} u^\alpha \in \mathrm{M}\big(\mathrm{c}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}(\mathbb{G}_{\max})\big)$.
- We have $(\widehat{\Delta} \otimes \mathrm{id})V = V_{23}V_{13}$.
- There is a notion of tensor product: $V \textcircled{\scriptsize{T}} U = V_{12}U_{13}$.
- Contragredient corepresentation: $V^{\mathrm{c}} = V^{\top \otimes \widehat{R}}$ ($\widehat{R}$ is the **unitary antipode** of $\widehat{\mathbb{G}}$ and $\top$ is the transposition).

DIGRESSION ON $L^2(\mathbb{G})$

- $\mathbb{G}$ has **Haar measure** — certain state h on $\mathrm{C}(\mathbb{G})$,
- $L^2(\mathbb{G})$ is the GNS space obtained from h,
- $L^2(\mathbb{G})$ has basis

$$\left\{u^{\alpha}_{i,j} \,\middle|\, \alpha \in \mathrm{Irr}(\mathbb{G}),\ i,j = 1, \ldots, n_{\alpha}\right\},$$

- there are interesting Peter-Weyl-Woronowicz orthogonality relations,
- we write $L^2(\mathbb{G})^{\alpha}$ for the subspace spanned by

$$\left\{u^{\alpha}_{i,j} \,\middle|\, i,j = 1, \ldots, n_{\alpha}\right\},$$

- $c_0(\widehat{\mathbb{G}}) = \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} M_{n_{\alpha}}(\mathbb{C})$ acts naturally on $L^2(\mathbb{G}) = \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} L^2(\mathbb{G})^{\alpha}$.

Property (T)

Definition (P. Fima, 2008)

- *A corepresentation* $V \in \mathrm{M}\big(\mathrm{c}_0(\widehat{\mathbb{G}}) \otimes \mathscr{K}(\mathscr{H})\big)$ *of* $\widehat{\mathbb{G}}$ ***has almost invariant vectors*** *if for any finite subset* $E \subset \mathrm{Irr}(\mathbb{G})$ *and any* $\delta > 0$ *there exists* $\xi \in \mathscr{H}$ *such that*

$$\big\|V^{\alpha}(\eta \otimes \xi) - \eta \otimes \xi\big\| < \delta \|\eta\| \|\xi\|$$

for all $\alpha \in E$ *and all* $\eta \in L^2(\mathbb{G})^{\alpha}$*.*

- $\widehat{\mathbb{G}}$ *has property* (T) *if every corepresentation* V *with almost invariant vectors has a non-zero invariant vector i.e. a non-zero* $\xi \in \mathscr{H}$ *such that*

$$V(\eta \otimes \xi) = \eta \otimes \xi$$

for all $\eta \in L^2(\mathbb{G})$*.*

Other characterizations

Theorem (David Kyed & P.M.S.)

The following are equivalent:

- $\widehat{\mathbb{G}}$ *has property* (T),
- *the counit* ϵ *is an isolated point of* $\mathrm{Spec}\big(\mathrm{C}(\mathbb{G}_{\max})\big)$,
- *all finite dimensional representations are isolated points of* $\mathrm{Spec}\big(\mathrm{C}(\mathbb{G}_{\max})\big)$,
- *the* C^**-algebra* $\mathrm{C}(\mathbb{G}_{\max})$ *has property* (T) *of Bekka,*
- *there exists a unique minimal projection* p *in the center of* $\mathrm{C}(\mathbb{G}_{\max})$ *with* $\epsilon(p) = 1$,
- *there exists a minimal projection* $p \in \mathrm{C}(\mathbb{G}_{\max})$ *with* $\epsilon(p) = 1$,
- $\widehat{\mathbb{G}}$ *has property* (T) *as defined by Petrescu & Joita (1992),*
- $\widehat{\mathbb{G}}$ *has property* (T) *as defined by Bédos, Conti & Tuset (2005).*

First exotic examples

Theorem

Take a non-coamenable $\mathbb{G}$.* *Then*

- $C(\mathbb{G}_{\mathrm{min}}) \neq C(\widetilde{\mathbb{G}_{\mathrm{min}}})$,
- *if* $C(\widetilde{\mathbb{G}_{\mathrm{min}}}) = C(\mathbb{G}_{\mathrm{max}})$ *then* $\widehat{\mathbb{G}}$ *has property* (T).

This provides many examples such that

$$\mathbb{G}_{\mathrm{min}} \neq \mathbb{G} \neq \mathbb{G}_{\mathrm{max}}$$

(take $\mathbb{G} = \widetilde{\mathbb{G}_{\mathrm{min}}}$ with $\mathbb{G}$ without property (T)).

*i.e. $C(\mathbb{G}_{\mathrm{min}}) \neq C(\mathbb{G}_{\mathrm{max}})$

Digression on corepresentations

Theorem (David Kyed & P.M.S.)

Let V and U be corepresentations of $\widehat{\mathbb{G}}$.

- *If there is a finite dimensional W such that $W \leq V$ and $W \leq U^{c}$ then the trivial representation is contained in $V \textcircled{\top} U$.*
- *If $\widehat{\mathbb{G}}$ is unimodular* and $V \textcircled{\top} U$ contains the trivial representation then there exists a finite dimensional W such that $W \leq V$ and $W \leq U^{c}$.*

Fact

Any discrete quantum group with property (T) is unimodular.

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*the Haar measure on $\mathbb{G}$ is a trace

Special representation

- Let Π be the representation of $\mathbb{G}_{\mathrm{max}}$ which is the direct sum of all infinite-dimensional irreducible representations.

Theorem

If $\widehat{\mathbb{G}}$ *has property* (T) *then the* C^**-norm on* $\mathrm{Pol}(\mathbb{G})$ *defined by* Π *is a quantum group norm.*

- Denote the resulting quantum group by $\mathbb{G}_{\Pi}$.

More exotic examples

- Take $\widehat{\mathbb{G}}$ an infinite property (T) discrete quantum group.
- $\mathbb{G}_\Pi$ does not admit a continuous counit, so

$$\mathbb{G}_\Pi \neq \widetilde{\mathbb{G}_\Pi}.$$

- It could happen that $\mathbb{G}_{\mathrm{min}} = \mathbb{G}_\Pi$, but in most cases

$$\mathbb{G}_{\mathrm{min}} \neq \mathbb{G}_\Pi.$$

- there are examples when $\widetilde{\mathbb{G}_\Pi} = \mathbb{G}_{\mathrm{max}}$, but in most cases

$$\widetilde{\mathbb{G}_\Pi} \neq \mathbb{G}_{\mathrm{max}}.$$

Summary

- $\mathbb{G}$ — coamenable

$$\mathbb{G}_{\text{min}} = \mathbb{G} = \widetilde{\mathbb{G}} = \mathbb{G}_{\text{max}}.$$

- $\mathbb{G}$ — non-coamenable, $\widehat{\mathbb{G}}$ not Kazhdan

$$\mathbb{G}_{\text{min}} = \mathbb{G} \neq \widetilde{\mathbb{G}} \neq \mathbb{G}_{\text{max}}.$$

- $\widehat{\mathbb{G}}$ — Kazhdan, minimally almost periodic

$$\mathbb{G}_{\text{min}} \neq \mathbb{G} \neq \widetilde{\mathbb{G}} = \mathbb{G}_{\text{max}}.$$

- $\widehat{\mathbb{G}}$ — Kazhdan, not minimally almost periodic

$$\mathbb{G}_{\text{min}} \neq \mathbb{G} \neq \widetilde{\mathbb{G}} \neq \mathbb{G}_{\text{max}}.$$