On the Heisenberg double

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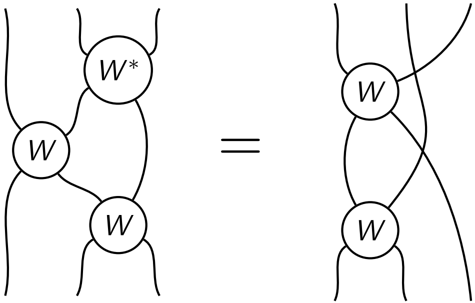
December 12, 2007

Outline of talk

Multiplicative unitaries

A unitary $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ is multiplicative if



on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$.

Modular multiplicative unitaries

A unitary $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ is a modular multiplicative unitary if

- W is multiplicative ($W_{23}W_{12} = W_{12}W_{13}W_{23}$)
- there exist positive, selfadjoint Q and $\widehat{Q}$ on $\mathcal{H}$ such that

$$W\big(\widehat{Q} \otimes Q\big)W^* = \widehat{Q} \otimes Q$$

- we have

$$\big(x \otimes y \,\big|\, W(z \otimes u)\big) = \big(\overline{z} \otimes Qy \,\big|\, \widetilde{W}(\overline{x} \otimes Q^{-1}u)\big)$$

for a certain unitary $\widetilde{W} \in \mathrm{B}\big(\overline{\mathcal{H}} \otimes \mathcal{H}\big)$

From l.c.q.g. to multiplicative unitaries

- Take a locally compact quantum group (A,Δ,φ,ψ)
- The right regular representation is the extension of

$$A\otimes A \ni (a\otimes b) \longmapsto \Delta(a)(\mathbb{1}\otimes b) \in A\otimes A$$

  to a unitary operator W on $\mathcal{H}_\psi\otimes\mathcal{H}_\psi \supset A\otimes A$
- This is a modular multiplicative unitary (with $\widehat{Q}=Q$)
- $W\in \mathrm{B}(\mathcal{H}_\psi\otimes\mathcal{H}_\psi)$ defines (A,Δ)

From multiplicative unitaries to quantum groups

- Take modular m.u. $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$
- Let $A = \left\{(\omega \otimes \mathrm{id})(W) \middle| \omega \in \mathrm{B}(\mathcal{H})_*\right\}^{\text{norm closure}} \subset \mathrm{B}(\mathcal{H})$
- A is a C^*-algebra
- For $a \in A$ we have $W(a \otimes \mathbb{1})W^* \in \mathrm{M}(A \otimes A)$ and

$$A \ni a \longmapsto W(a \otimes \mathbb{1})W^*$$

defines a comultiplication $\Delta \in \mathrm{Mor}(A, A \otimes A)$.

Quantum group from W

- We have $(\Delta\otimes\mathrm{id})\circ\Delta=(\mathrm{id}\otimes\Delta)\circ\Delta$
- We have

$$\mathrm{span}\left\{\Delta(a)(\mathbb{1}\otimes b)\,\middle|\,a,b\in A\right\}\subset_{\text{dense}}A\otimes A$$
$$\mathrm{span}\left\{(a\otimes\mathbb{1})\Delta(b)\,\middle|\,a,b\in A\right\}\subset_{\text{dense}}A\otimes A$$

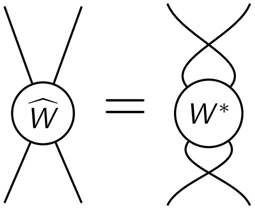
- There is a closed antimultiplicative map

$$\kappa:(\omega\otimes\mathrm{id})(W)\longmapsto(\omega\otimes\mathrm{id})(W^*)$$

- Moreover $\kappa=R\circ\tau_{\frac{i}{2}}$ where R is an antiautomorphism of A and $(\tau_t)_{t\in\mathbb{R}}$ is a one parameter group of automorphisms of A: $\tau_t(a)=Q^{2it}aQ^{-2it}$

Further structure: reduced dual I

- Let $\widehat{W} = \Sigma W^* \Sigma$, i.e.



- $\widehat{W}$ is a modular multiplicative unitary.
- Thus we get another quatum group $\left(\widehat{A}, \widehat{\Delta}\right)$

Further structure: reduced dual II

We have

- $\widehat{A} = \left\{(\mathrm{id} \otimes \omega)(W) \middle| \omega \in \mathrm{B}(\mathcal{H})_*\right\}^{\text{norm closure}} \subset \mathrm{B}(\mathcal{H})$
- $W \in \mathrm{M}(\widehat{A} \otimes A)$
- Reduced dual of $(\widehat{A}, \widehat{\Delta})$ is (A, Δ)
- We have $\widehat{\kappa}$, $\widehat{R}$, $(\widehat{\tau}_t)_{t \in \mathbb{R}}$: $\widehat{\tau}(\widehat{a}) = \widehat{Q}^{2it}\widehat{a}\widehat{Q}^{-2it}$

The problem

Fact

There can be different modular multiplicative unitary operators $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ giving rise to the same (A, Δ).

Theorem

1. *The maps κ, R and the group $(\tau_t)_{t \in \mathbb{R}}$ are independent of the choice of W and $\mathcal{H}$.*
2. *The reduced dual $(\widehat{A}, \widehat{\Delta})$ (with all its structure) is independent of the choice of W and $\mathcal{H}$.*
3. *The position of W in $\mathrm{M}(\widehat{A} \otimes A)$ is independent of the choice of W and $\mathcal{H}$ (thus we get $\mathbb{V} \in \mathrm{M}(\widehat{A} \otimes A)$).*
4. *The ultraweak topology on A from embedding into $\mathrm{B}(\mathcal{H})$ is independent of the choice of W and $\mathcal{H}$.*

The blessing

Theorem

If $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ is a m.m.u. giving rise to (A, Δ) then the weight

$$\psi_W : A_+ \ni a \longmapsto \mathrm{Tr}\left(\widehat{Q} a \widehat{Q}\right) \in [0, \infty]$$

is right invariant. It is the Haar measure of (A, Δ) if it is locally finite.

Thus

we can look for the Haar measure of (A, Δ) by examining ψ_W for all possible W's.

Heisenberg representations

- We fix (A, Δ) and thus $(\widehat{A}, \widehat{\Delta})$ with $\mathbb{V} \in \mathrm{M}(\widehat{A} \otimes A)$.
- A pair of representations $(\widehat{\pi}, \pi)$ of $\widehat{A}$ and A on a Hilbert space H is called a Heisenberg pair if

$$\mathbb{V}_{\widehat{\pi}3}\mathbb{V}_{1\pi}\mathbb{V}^{*}_{\widehat{\pi}3} = \mathbb{V}_{1\pi}\mathbb{V}_{13}$$

  in $\mathrm{M}(\widehat{A} \otimes \mathcal{K}(H) \otimes A)$
- (where $\mathbb{V}_{\widehat{\pi}3} = (\mathrm{id} \otimes \widehat{\pi} \otimes \mathrm{id})\mathbb{V}_{23}$, $\mathbb{V}_{1\pi} = (\mathrm{id} \otimes \pi \otimes \mathrm{id})\mathbb{V}_{12}$)
- We get such a pair from every m.u. giving rise to (A, Δ)

C^*-algebra of a Heisenberg pair

- With every Heisenberg pair $(\widehat{\pi},\pi)$ we associate the subspace

$$D_{\widehat{\pi},\pi}=\overline{\widehat{\pi}(\widehat{A})\pi(A)}\subset \mathrm{B}(H).$$

- This is a C^*-algebra:
  - $\mathbb{V}^*_{1\pi}\mathbb{V}_{\widehat{\pi}3}=\mathbb{V}_{13}\mathbb{V}_{\widehat{\pi}3}\mathbb{V}^*_{1\pi}$
  - $A=\left\{(\omega\otimes\mathrm{id})(\mathbb{V})\,\middle|\,\omega\in\widehat{A}_*\right\}^{\text{norm closure}}$
  - $\widehat{A}=\left\{(\mathrm{id}\otimes\omega)(\mathbb{V}^*)\,\middle|\,\omega\in A_*\right\}^{\text{norm closure}}$
- $\pi\in\mathrm{Mor}(A,D_{\widehat{\pi},\pi})$, $\widehat{\pi}\in\mathrm{Mor}\left(\widehat{A},D_{\widehat{\pi},\pi}\right)$
- $D_{\widehat{\pi},\pi}$ can be also seen as closure of image of $\widehat{A}\otimes_{\mathrm{alg}}A$ under the map

$$\sum a_i\otimes b_i\longmapsto\sum\widehat{\pi}(a_i)\pi(b_i)$$

The universal algebra for Heisenberg relations

- For $x \in \widehat{A} \otimes_{\mathrm{alg}} A$ let

$$\|x\| = \sup_{(\widehat{\pi},\pi)} \left\|(\widehat{\pi} \otimes \pi)(x)\right\|.$$

- There is a Heisenberg pair $(\widehat{\rho}, \rho)$ realizing the supremum
- We define the Heisenberg double D of (A, Δ)

$$D := D_{\widehat{\rho},\rho}$$

- We have $\rho \in \mathrm{Mor}(A, D)$, $\widehat{\rho} \in \mathrm{Mor}\big(\widehat{A}, D\big)$ and we can treat A and $\widehat{A}$ as subalgebras of $\mathrm{M}(D)$, $\overline{\widehat{A}A} = D$

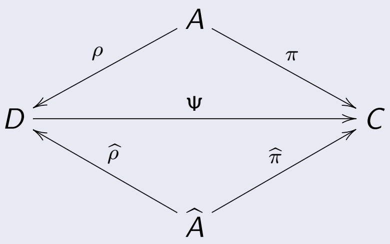
Properties of D I

Theorem

For any C^**-algebra* C *and* $\pi \in \mathrm{Mor}(A, C)$, $\widehat{\pi} \in \mathrm{Mor}\big(\widehat{A}, C\big)$ *such that*

$$\mathbb{V}_{\widehat{\pi}3}\mathbb{V}_{1\pi}\mathbb{V}^*_{\widehat{\pi}3} = \mathbb{V}_{1\pi}\mathbb{V}_{13}$$

then there exists a unique $\Psi \in \mathrm{Mor}(D, C)$ *such that*



Properties of D II

Theorem

1. *There is a unique* $\widehat{\Phi} \in \mathrm{Mor}\big(D, \widehat{A} \otimes D\big)$ *such that*
   - $\widehat{\Phi}(a) = \mathbb{1} \otimes a$ *for* $a \in A \subset \mathrm{M}(D)$
   - $\widehat{\Phi}(\widehat{a}) = \widehat{\Delta}(\widehat{a})$ *for* $\widehat{a} \in \widehat{A} \subset \mathrm{M}(D)$
2. *There is a unique* $\Phi \in \mathrm{Mor}(D, A \otimes D)$ *such that*
   - $\Phi(a) = \Delta(a)$ *for* $a \in A$
   - $\Phi(\widehat{a}) = \mathbb{1} \otimes \widehat{a}$ *for* $\widehat{a} \in \widehat{A}$
3. $\widehat{\Phi}$ *and* Φ *are continuous left actions*
   - $(\mathrm{id} \otimes \widehat{\Phi}) \circ \widehat{\Phi} = (\widehat{\Delta} \otimes \mathrm{id}) \circ \widehat{\Phi}$, $(\mathrm{id} \otimes \Phi) \circ \Phi = (\Delta \otimes \mathrm{id}) \circ \Phi$
   - $\overline{\widehat{\Phi}(D)\big(\widehat{A} \otimes \mathbb{1}\big)} = \widehat{A} \otimes D$, $\overline{\Phi(D)\big(A \otimes \mathbb{1}\big)} = A \otimes D$

Properties of D III

Sketch of proof.

To define $\widehat{\Phi}$ we use universal property of D for

$$\pi(a) = \mathbb{1} \otimes a$$
$$\widehat{\pi}(\widehat{a}) = \widehat{\Delta}(\widehat{a})$$

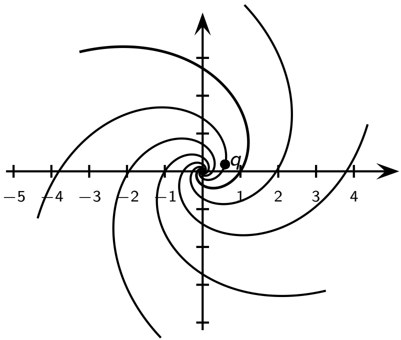
Also $D = \overline{\widehat{A}A} = \overline{A\widehat{A}}$, so

$$\begin{aligned}\widehat{\Phi}(D)(\widehat{A} \otimes \mathbb{1}) &= \widehat{\Phi}(A)\widehat{\Phi}(\widehat{A})(\widehat{A} \otimes \mathbb{1}) = (\mathbb{1} \otimes A)\widehat{\Delta}(\widehat{A})(\widehat{A} \otimes \mathbb{1}) \\ &= (\mathbb{1} \otimes A)(\widehat{A} \otimes \widehat{A}) = (\widehat{A} \otimes A\widehat{A}) = \widehat{A} \otimes D\end{aligned}$$

up to closure. □

Quantum "$az+b$" group I

- (A,Δ) generated by $a, a^{-1}, b\,\eta\,A$, $ab = q^2 ba$, $ab^* = b^* a$ and
- spectral conditions on a and b:



Quantum "$az+b$" group II

- $\Delta(a) = a \otimes a$
- $\Delta(b) = a \otimes b \dot{+} b \otimes \mathbb{1}$
- There is a m.m.u. $W_{\mathrm{s}} \in \mathrm{B}(K \otimes K)$ giving rise to (A, Δ) with

$$A'' = \widehat{A}''$$
$$W_{\mathrm{s}} = F\big(ab^{-1} \otimes b\big)\chi\big(b^{-1} \otimes \mathbb{1}, \mathbb{1} \otimes a\big)$$

- We know general form of all representations of (A, Δ):
  - $\mathcal{V} \in \mathrm{M}\big(\mathcal{K}(H) \otimes A\big)$, $(\mathrm{id} \otimes \Delta)(\mathcal{V}) = \mathcal{V}_{12}\mathcal{V}_{13}$
  - $\mathcal{V} = F\big(\widehat{b} \otimes b\big)\chi\big(\widehat{a} \otimes \mathbb{1}, \mathbb{1} \otimes a\big)$

(F and χ are certain special functions)

Multiplicative unitaries for (A, Δ)

Theorem

Let $U \in \mathrm{B}(H \otimes H)$ be a m.u. giving rise to the quantum "$az+b$" group (A, Δ). Then

$$U = W_s V,$$

where $V = F\big(\widehat{b} \otimes b\big)\chi\big(\widehat{a} \otimes \mathbb{1}, \mathbb{1} \otimes a\big)$, where $\big(\widehat{a}, \widehat{b}\big)$ commute with (a, b).
U can be also written as

$$U = F\big((a \dotplus \widehat{b})b^{-1} \otimes b\big)\chi\big(\widehat{a}b^{-1} \otimes \mathbb{1}, \mathbb{1} \otimes a\big)$$

This says that $(\mathrm{id} \otimes \pi)\widehat{\Phi}$ is faithful (π given by U).