A personal view of compact and non-compact quantum groups

Operator Algebra Seminar

Seoul National University

Piotr M. Sołtan

Department of Mathematical Methods in Physics
Faculty of Physics, University of Warsaw

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What to do with a $q \in [-1,1]$?

- Let us say that two elements $\dot{\alpha}$ and $\dot{\gamma}$ of a unital $*$-algebra **satisfy** $\mathrm{SU}_q(2)$**-relations** if

$$\begin{gathered}\dot{\alpha}\dot{\gamma} = q\dot{\gamma}\dot{\alpha}, \qquad \dot{\gamma}^*\dot{\gamma} = \dot{\gamma}\dot{\gamma}^*, \qquad \dot{\alpha}\dot{\gamma}^* = q\dot{\gamma}^*\dot{\alpha},\\ \dot{\alpha}^*\dot{\alpha} + \dot{\gamma}^*\dot{\gamma} = \mathbb{1}, \qquad \dot{\alpha}\dot{\alpha}^* + q^2\dot{\gamma}^*\dot{\gamma} = \mathbb{1}.\end{gathered}$$

- Let $\mathcal{F}$ be the free unital $*$-algebra generated by symbols $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$.
- For $x \in \mathcal{F}$ let $\|x\| = \sup\bigl\|\pi(x)\bigr\|$ with supremum over unital $*$-homomorphisms into $\mathrm{B}(\mathcal{H})$ such that $\pi(\boldsymbol{\alpha})$ and $\pi(\boldsymbol{\gamma})$ satisfy $\mathrm{SU}_q(2)$-relations.
- $\|\cdot\|$ is a C*-seminorm on $\mathcal{F}$ and we let A_q be the completion of $\mathcal{F}/\bigl\{x \,\big|\, \|x\| = 0\bigr\}$.
- Let α and γ be images of $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ in A_q. Clearly they satisfy $\mathrm{SU}_q(2)$-relations.
- If B is a unital C*-algebra with $\bar{\alpha}, \bar{\gamma} \in \mathsf{B}$ satisfying $\mathrm{SU}_q(2)$-relations then there exist a unique unital $*$-homomorphism $\mathsf{A}_q \to \mathsf{B}$ mapping α to $\bar{\alpha}$ and γ to $\bar{\gamma}$.

Remark

Elements $\dot{\alpha}$ and $\dot{\gamma}$ of a a unital $*$-algebra $\mathscr{B}$ satisfy $\mathrm{SU}_q(2)$-relations if and only if $\begin{bmatrix} \dot{\alpha} & -q\dot{\gamma}^* \\ \dot{\gamma} & \dot{\alpha}^* \end{bmatrix} \in \mathsf{Mat}_2(\mathscr{B})$ is unitary.

- In particular $U = \begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix} \in \mathsf{Mat}_2(\mathsf{A}_q)$ is unitary.
- Let $U' = \begin{bmatrix} \alpha\otimes\mathbb{1} & -q\gamma^*\otimes\mathbb{1} \\ \gamma\otimes\mathbb{1} & \alpha^*\otimes\mathbb{1} \end{bmatrix}$ and $U'' = \begin{bmatrix} \mathbb{1}\otimes\alpha & -q\mathbb{1}\otimes\gamma^* \\ \mathbb{1}\otimes\gamma & \mathbb{1}\otimes\alpha^* \end{bmatrix}$. Clearly U' and U'' are unitary elements of $\in \mathsf{Mat}_2(\mathsf{A}_q \otimes \mathsf{A}_q)$.
- Hence so is their product $\begin{bmatrix} \alpha\otimes\alpha - q\gamma^*\otimes\gamma & -q\gamma^*\otimes\alpha^* - q\alpha\otimes\gamma^* \\ \gamma\otimes\alpha + \alpha^*\otimes\gamma & \alpha^*\otimes\alpha^* - q\gamma\otimes\gamma^* \end{bmatrix}$.
- It follows that there is a unique $\Delta\colon \mathsf{A}_q \to \mathsf{A}_q \otimes \mathsf{A}_q$ such that

$$\Delta(\alpha) = \alpha\otimes\alpha - q\gamma^*\otimes\gamma, \qquad \Delta(\gamma) = \gamma\otimes\alpha + \alpha^*\otimes\gamma.$$

- Identifying $\mathsf{Mat}_2(\mathsf{A}_q)$ with $\mathsf{Mat}_2(\mathbb{C}) \otimes \mathsf{A}_q$ we can write $(\mathrm{id}\otimes\Delta)U = U'U''$.
- Using this it is easy to check that $(\Delta\otimes\mathrm{id})\circ\Delta = (\mathrm{id}\otimes\Delta)\circ\Delta$.

- Let

$$V=\begin{bmatrix} v_{1,1} & \cdots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{n,1} & \cdots & v_{n,n} \end{bmatrix} \in \mathsf{Mat}_n(\mathsf{A}_q) \cong \mathsf{Mat}_n(\mathbb{C}) \otimes \mathsf{A}_q$$

be a unitary matrix such that $(\mathrm{id} \otimes \Delta)V = V'V''$, where

$$V'=\begin{bmatrix} v_{1,1}\otimes \mathbb{1} & \cdots & v_{1,n}\otimes \mathbb{1} \\ \vdots & \ddots & \vdots \\ v_{n,1}\otimes \mathbb{1} & \cdots & v_{n,n}\otimes \mathbb{1} \end{bmatrix} \quad \text{and} \quad V''=\begin{bmatrix} \mathbb{1}\otimes v_{1,1} & \cdots & \mathbb{1}\otimes v_{1,n} \\ \vdots & \ddots & \vdots \\ \mathbb{1}\otimes v_{n,1} & \cdots & \mathbb{1}\otimes v_{n,n} \end{bmatrix}$$

- Then $\big((\mathrm{id} \otimes \Delta)V\big)V''^{*} = V'$, so for any (k,l) and any $b \in \mathsf{A}_q$ the element $v_{k,l} \otimes b$ belongs to the subspace

$$\mathrm{span}\big\{\Delta(a)(\mathbb{1} \otimes b)\,\big|\, a, b \in \mathsf{A}_q\big\}.$$

- How to construct unitaries $V \in \mathsf{Mat}_n(\mathsf{A}_q)$ such that $(\mathrm{id} \otimes \Delta)V = V'V''$?
- For any two such matrices $V \in \mathsf{Mat}_n(\mathsf{A}_q)$ and $W \in \mathsf{Mat}_m(\mathsf{A}_q)$ we can form $V \oplus W \in \mathsf{Mat}_{nm}(\mathsf{A}_q)$:

$$V \oplus W = \sum_{i,j,k,l} e^n_{i,j} \otimes e^m_{k,l} \otimes v_{i,j} w_{k,l}.$$

- One can check that $V \oplus W$ is again a unitary and

$$(\mathrm{id} \otimes \Delta)(V \oplus W) = (V \oplus W)'(V \oplus W)''$$

  as before.
- Now start with $V = W = U \in \mathsf{Mat}_2(\mathsf{A}_q)$ and continue...
- One can show that **if** $q \neq 0$ then any monomial in α, γ, α^* and γ^* appears as an element of one of these matrices.

Corollary

The sets

$$\operatorname{span}\left\{\Delta(a)(\mathbb{1}\otimes b)\,\middle|\,a,b\in\mathsf{A}_q\right\}\quad\text{and}\quad\operatorname{span}\left\{(a\otimes\mathbb{1})\Delta(b)\,\middle|\,a,b\in\mathsf{A}_q\right\}$$

are dense in $\mathsf{A}_q\otimes\mathsf{A}_q$.

Definition

A **compact quantum group** is an object described by a unital C*-algebra A together with a unital $*$-homomorphism $\Delta\colon\mathsf{A}\to\mathsf{A}\otimes\mathsf{A}$ such that

1. $(\Delta\otimes\mathrm{id})\circ\Delta=(\mathrm{id}\otimes\Delta)\circ\Delta$,
2. The sets

$$\operatorname{span}\left\{\Delta(a)(\mathbb{1}\otimes b)\,\middle|\,a,b\in\mathsf{A}\right\}\quad\text{and}\quad\operatorname{span}\left\{(a\otimes\mathbb{1})\Delta(b)\,\middle|\,a,b\in\mathsf{A}\right\}$$

are dense in $\mathsf{A}\otimes\mathsf{A}$.

- A "q-deformation" analogous to the passage $\mathrm{SU}(2) \rightsquigarrow \mathrm{SU}_q(2)$ can be performed for any semisimple compact Lie group G.
- Briefly:
  - we write the Serre presentation of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G,
  - we cleverly introduce the deformation parameter q into the relations so that the resulting algebra $\mathcal{U}(\mathfrak{g})$ retains the structure of a Hopf $*$-algebra,
  - we consider a certain category $\mathcal{C}$ of modules over the resulting algebra and find that it is a rigid C^*-tensor category,
  - we use the **Tannaka-Krein-Woronowicz** theorem to construct a compact quantum group G_q such that $\mathcal{C}$ is the category of representations of G_q.
- Later we will see that "quantum analogs" of some non-compact groups can also be constructed.

Notation

Typically a quantum group will be denoted by a symbol like $\mathbb{G}$. Then the corresponding C^*-algebra is denoted by $\mathrm{C}(\mathbb{G})$.

- The compact quantum group defined by (A_q, Δ) on previous slides is called $\mathrm{SU}_q(2)$, so instead of A_q we usually write $\mathrm{C}(\mathrm{SU}_q(2))$.
- When $q = 1$ the algebra $\mathrm{C}(\mathrm{SU}_q(2))$ is commutative and isomorphic to $\mathrm{C}(\mathrm{SU}(2))$. The comultiplication becomes the map

$$\Delta\colon\ \mathrm{C}(\mathrm{SU}(2)) \longrightarrow \mathrm{C}(\mathrm{SU}(2)) \otimes \mathrm{C}(\mathrm{SU}(2)) \cong \mathrm{C}(\mathrm{SU}(2) \times \mathrm{SU}(2))$$

  given by $\Delta(f)(x, y) = f(xy)$ for all $f \in \mathrm{C}(\mathrm{SU}(2))$ and all $x, y \in \mathrm{SU}(2)$.
- When $q \neq 1$ the algebra $C(\mathrm{SU}_q(2))$ is not commutative.

Theorem

Let $\mathbb{G}$ be a compact quantum group. If $\mathrm{C}(\mathbb{G})$ is commutative then the compact space G such that $\mathrm{C}(\mathbb{G}) \cong \mathrm{C}(G)$ carries a unique group structure such that

$$\Delta\colon \mathrm{C}(G) \longrightarrow \mathrm{C}(G) \otimes \mathrm{C}(G) \cong \mathrm{C}(G \times G)$$

satisfies $\Delta(f)(x,y) = f(xy)$ for all $f \in \mathrm{C}(G)$ and all $x, y \in G$.

- In case $\mathrm{C}(\mathbb{G}) = \mathrm{C}(G)$ is commutative, the condition that

$$\operatorname{span}\left\{(a \otimes \mathbb{1})\Delta(b) \,\middle|\, a, b \in \mathsf{A}\right\} \quad \text{and} \quad \operatorname{span}\left\{\Delta(a)(\mathbb{1} \otimes b) \,\middle|\, a, b \in \mathsf{A}\right\}$$

are dense in $\mathrm{C}(G) \otimes \mathrm{C}(G)$ translates to **cancellation laws**: $xy = xz \;\Rightarrow\; y = z$ and $xy = zy \;\Rightarrow\; x = z$.

Example

Let Γ be a discrete group. Recall that $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ is the C^*-algebra of operators on $\mathrm{B}(\ell_2(\Gamma))$ generated by operators λ_t ($t \in \Gamma$) of the left regular representation. There exists a unique $\Delta\colon \mathrm{C}^*_{\mathrm{r}}(\Gamma) \to \mathrm{C}^*_{\mathrm{r}}(\Gamma) \otimes \mathrm{C}^*_{\mathrm{r}}(\Gamma)$ such that

$$\Delta(\lambda_t) = \lambda_t \otimes \lambda_t, \qquad t \in \Gamma$$

and it is not hard to see that $\mathbb{G}$ defined by $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ and Δ as above is a compact quantum group. It is usually denoted by $\widehat{\Gamma}$.

THEOREM (S.L. WORONOWICZ)

Let $\mathbb{G}$ be a compact quantum group. Then there exists a unique state $\boldsymbol{h}$ on $\mathrm{C}(\mathbb{G})$ such that

$$(\boldsymbol{h}\otimes\mathrm{id})\Delta(a)=\boldsymbol{h}(a)\mathbb{1}=(\mathrm{id}\otimes\boldsymbol{h})\Delta(a)$$

for all $a\in\mathrm{C}(\mathbb{G})$.

- The state $\boldsymbol{h}$ is called the **Haar measure** (or sometimes the **Haar state**) of $\mathbb{G}$.
- If $\mathbb{G}=G$ is classical then $\boldsymbol{h}$ is given by integration with respect to the normalized Haar measure on G.
- If $\mathbb{G}=\widehat{\Gamma}$ for some discrete group Γ then $\boldsymbol{h}$ is the von Neumann trace:

$$\mathrm{C}^*_{\mathrm{r}}(\Gamma)\ni x\longmapsto\langle\delta_e|x\delta_e\rangle\in\mathbb{C},$$

  where $\delta_e\in\ell_2(\Gamma)$ is the indicator function of $e\in\Gamma$.

Definition

Let $\mathbb{G}$ be a compact quantum group and let $\mathcal{H}$ be a finite dimensional Hilbert space. A **representation** of $\mathbb{G}$ on $\mathcal{H}$ is a unitary $U \in \mathrm{B}(\mathcal{H}) \otimes \mathrm{C}(\mathbb{G})$ such that

$$(\mathrm{id} \otimes \Delta)U = U_{12}U_{13},$$

where $U_{12} = U \otimes \mathbb{1} \in \mathrm{B}(\mathcal{H}) \otimes \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$ and U_{13} is the image of U under the map

$$\mathrm{B}(\mathcal{H}) \otimes \mathrm{C}(\mathbb{G}) \ni \boldsymbol{a} \otimes \boldsymbol{b} \mapsto \boldsymbol{a} \otimes \mathbb{1} \otimes \boldsymbol{b} \in \mathrm{B}(\mathcal{H}) \otimes \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G}).$$

- A choice of a basis in $\mathcal{H}$ allows us to write U as a matrix $U = \begin{bmatrix} u_{1,1} & \cdots & u_{1,n} \\ \vdots & \ddots & \vdots \\ u_{n,1} & \cdots & u_{n,n} \end{bmatrix}$.
- The $\{u_{i,j}\}$ are called the **matrix elements** of U.
- The condition $(\mathrm{id} \otimes \Delta)U = U_{12}U_{13}$ translates to $\Delta(u_{i,j}) = \sum\limits_{k=1}^{n} u_{i,k} \otimes u_{k,j}$.

- A representation U of $\mathbb{G}$ on $\mathcal{H}$ is **irreducible** if there does not exits a non-trivial projection $P \in \mathrm{B}(\mathcal{H})$ such that $(P \otimes \mathbb{1})U = U(P \otimes \mathbb{1})$.
- Two representations $U \in \mathrm{B}(\mathcal{H}) \otimes \mathrm{C}(\mathbb{G})$ and $V \in \mathrm{B}(\mathcal{K}) \otimes \mathrm{C}(\mathbb{G})$ are **equivalent** if there is a unitary $T \in \mathrm{B}(\mathcal{H}, \mathcal{K})$ such that $(T \otimes \mathbb{1})U = V(T \otimes \mathbb{1})$.
- We let $\mathrm{Irr}\,\mathbb{G}$ denote the set of equivalence classes of irreducible representations of $\mathbb{G}$. For $\alpha \in \mathrm{Irr}\,\mathbb{G}$ we fix $U^\alpha \in \alpha$ and a basis of its carrier space thus fixing the matrix elements $\{u^\alpha_{i,j}\}$.
- The matrix Q whose (i,j)-entry is $\sum\limits_k \boldsymbol{h}\big(u^\alpha_{k,j} {u^\alpha_{k,i}}^*\big)$ is positive and invertible. We put

$$\rho_\alpha = \sqrt{\frac{\mathrm{Tr}(Q^{-1})}{\mathrm{Tr}(Q)}}\,Q.$$

Theorem (S.L. Woronowicz)

1. *The span of* $\{u^{\alpha}_{i,j}\}$ *is a dense unital* $*$*-subalgebra of* $\mathrm{C}(\mathbb{G})$.
2. *For any* $\alpha, \beta, i, j, k, l$ *we have*

$$\boldsymbol{h}\bigl(u^{\alpha}_{k,l}{u^{\beta}_{i,j}}^{*}\bigr) = \delta_{\alpha,\beta}\delta_{k,i}\frac{(\rho_{\alpha})_{j,l}}{\mathrm{Tr}(\rho_{\alpha})} \quad \textit{and} \quad \boldsymbol{h}\bigl({u^{\beta}_{i,j}}^{*}u^{\alpha}_{k,l}\bigr) = \delta_{\alpha,\beta}\delta_{j,l}\frac{(\rho_{\alpha}^{-1})_{k,i}}{\mathrm{Tr}(\rho_{\alpha})}$$

- The above is a far reaching generalization of the Peter-Weyl theorem for compact groups.
- One can also make sense of the notion of a strongly continuous unitary representation of a compact quantum group on a Hilbert space of arbitrary dimension and show that those decompose as direct sums of finite dimensional irreducible ones.

Theorem (S.L. Woronowicz)

The following conditions are equivalent for a compact quantum group $\mathbb{G}$:

1. *The Haar measure* $\boldsymbol{h}$ *is a trace,*
2. $\mathrm{Tr}(\rho_\alpha) = \dim U^\alpha$ *for all* α*,*
3. $\rho_\alpha = \mathbb{1}$ *for all* α*.*

- Compact quantum groups with tracial Haar measures are referred to as being **of Kac type**.
- Using this notion one can show e.g. that the Toepliz algebra (a.k.a. the **quantum disk**) does not carry a compact quantum group structure (J. Krajczok+P.M.S.).
- A compact quantum group whose dimensions of irreducible representations are all bounded by some constant must be of Kac type (J. Krajczok+P.M.S.).

- Let $\mathbb{G}$ be a compact quantum group.
- Consider the element

$$W = \bigoplus_{\alpha \in \operatorname{Irr}\mathbb{G}} U^{\alpha} \in \bigoplus_{\alpha \in \operatorname{Irr}\mathbb{G}} \bigl(\mathrm{B}(\mathcal{H}^{\alpha}) \otimes \mathrm{C}(\mathbb{G})\bigr) \subset \mathrm{M}\Biggl(\Bigl(\bigoplus_{\alpha \in \operatorname{Irr}\mathbb{G}} \mathrm{B}(\mathcal{H}^{\alpha})\Bigr) \otimes \mathrm{C}(\mathbb{G})\Biggr),$$

  where $\mathrm{M}(\cdot)$ denotes the *multiplier algebra*.
- Thus letting $\mathrm{c}_0(\widehat{\mathbb{G}})$ denote the (non-unital) C^*-algebra $\bigoplus_{\alpha \in \operatorname{Irr}\mathbb{G}} \mathrm{B}(\mathcal{H}^{\alpha})$ we have

$$W \in \mathrm{M}(\mathrm{c}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}(\mathbb{G})).$$

- Furthermore $(\mathrm{id} \otimes \Delta)W = W_{12}W_{13}$.
- We will encounter this formula again, later on.

Back to $\mathrm{SU}_q(2)$

- The C*-algebra $\mathrm{C}(\mathrm{SU}_q(2))$ can be faithfully represented on $\ell_2(\mathbb{Z}_+ \times \mathbb{Z})$ so that

$$\alpha e_{i,j} = \sqrt{1-q^{2i}}\, e_{i-1,j}, \qquad \gamma e_{i,j} = q^i e_{i,j+1}.$$

- The Haar measure in this representation is given by

$$\boldsymbol{h}(a) = (1-q^2)^{-1} \sum_{i=0}^{\infty} q^{2i} \langle e_{i,0} \,|\, a e_{i,0} \rangle, \qquad a \in \mathrm{C}(\mathrm{SU}_q(2)).$$

- Consequently the GNS triple $(\mathcal{H}_{\boldsymbol{h}}, \pi_{\boldsymbol{h}}, \Omega_{\boldsymbol{h}})$ for $\boldsymbol{h}$ is:
  - $\mathcal{H}_{\boldsymbol{h}} = \bigoplus_{n=0}^{\infty} \ell_2(\mathbb{Z}_+ \times \mathbb{Z})$,
  - $\pi_{\boldsymbol{h}}(a) = (a, a, \dots)$ for all $a \in \mathrm{C}(\mathrm{SU}_q(2))$,
  - $\Omega_{\boldsymbol{h}} = \big(q^0 e_{0,0}, q^1 e_{1,0}, q^2 e_{2,0}, \dots\big)$.

A bigger Hilbert space

- The Hilbert space $\ell_2(\mathbb{Z}_+ \times \mathbb{Z})$ is naturally a subspace of $\ell_2(\mathbb{Z} \times \mathbb{Z})$.
- Consider the two operators v and n on $\ell_2(\mathbb{Z} \times \mathbb{Z})$:

$$ve_{i,j} = e_{i-1,j}, \qquad ne_{i,j} = q^i e_{i,j+1}.$$

- Clearly v is a unitary operator.
- Some care must be taken with n, as it is clearly not a bounded operator. We note that $e_{i,j} \mapsto q^i e_{i,j+1}$ is well defined and closable on the subspace $\operatorname{span}\{e_{i,j} \,|\, i,j \in \mathbb{Z}\}$ and we define n as the closure of this operator.
- It follows that n is a normal operator and $\operatorname{Sp}(n) = \left\{z \in \mathbb{C} \,\middle|\, |z| \in q^{\mathbb{Z}}\right\} \cup \{0\}$.
- One easily checks that the operators v and n satisfy the relation: $vnv^* = qn$.
- The statements in blue are the analog of $\mathrm{SU}_q(2)$-relations for α and γ.

The algebra generated by υ and n

- Let B_q be the closure in $\mathrm{B}(\ell_2(\mathbb{Z}\times\mathbb{Z}))$ of the set

$$\left\{\sum\nolimits' \upsilon^k f_k(n)\,\middle|\, f_k\in \mathrm{C}_0(\mathrm{Sp}(n))\right\}$$

  (with $\sum'$ meaning a finite sum).
- Then B_q is a non-degenerate (non-unital) C^*-subalgebra of $\mathrm{B}(\ell_2(\mathbb{Z}\times\mathbb{Z}))$.
- Moreover, υ belongs to the multiplier algebra $\mathrm{M}(\mathsf{B}_q)$ and n is **affiliated** with B_q which means that $n(\mathbb{1}+n^*n)^{-1/2}\in\mathrm{M}(\mathsf{B}_q)$ and $\overline{(\mathbb{1}+n^*n)^{-1/2}\mathsf{B}_q}=\mathsf{B}_q$.
- In particular, for any C^*-algebra C and a non-degenerate $*$-homomorphism $\Phi\colon\mathsf{B}_q\to\mathsf{C}$ it makes sense to write $\Phi(\upsilon)$ and $\Phi(n)$. Furthermore, $\Phi(\upsilon)\in\mathrm{M}(\mathsf{C})$ and $\Phi(n)$ is affiliated with C.

The quantum group $\mathrm{E}_q(2)$

- The operator $v \otimes n + n \otimes v^*$ defined on $\mathrm{Dom}(n) \odot \mathrm{Dom}(n)$ is closable and we denote its closure by $v \otimes n \dot{+} n \otimes v^*$.
- The C*-algebra $\mathsf{B}_q \otimes \mathsf{B}_q$ is naturally represented on $\ell_2(\mathbb{Z} \times \mathbb{Z}) \otimes \ell_2(\mathbb{Z} \times \mathbb{Z})$.

Theorem

There exists a unique non-degenerate $*$-homomorphism $\Delta \colon \mathsf{B}_q \to \mathsf{B}_q \otimes \mathsf{B}_q$ such that

$$\Delta(v) = v \otimes v \quad \textit{and} \quad \Delta(n) = v \otimes n \dot{+} n \otimes v^*.$$

Moreover $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and the sets

$$\mathrm{span}\{\Delta(a)(\mathbb{1} \otimes b) \,|\, a, b \in \mathsf{B}_q\} \quad \textit{and} \quad \mathrm{span}\{(a \otimes \mathbb{1})\Delta(b) \,|\, a, b \in \mathsf{B}_q\}$$

are dense in $\mathsf{B}_q \otimes \mathsf{B}_q$.

The quantum group $\mathrm{E}_q(2)$

- Let F_q be the continuous function on $\mathrm{Sp}(n)$ defined by

$$F_q(z) = \prod_{n=0}^{\infty} \frac{1+q^{2n}\overline{z}}{1+q^{2n}z}, \qquad z \notin \{-1, -q^2, -q^4, \dots\}$$

- The operator $n^{-1}v \otimes vn$ is normal and $\mathrm{Sp}(n^{-1}v \otimes vn) = \mathrm{Sp}(n)$.

Theorem

Let $W = F_q(n^{-1}v \otimes vn)V$, where V is the unitary operator $e_{i,j} \otimes e_{k,l} \mapsto e_{i,j} \otimes e_{k+i+j,l}$. Then W is a unitary operator and

$$\Delta(a) = W(a \otimes \mathbb{1})W^*, \qquad a \in \mathsf{B}_q.$$

Moreover $W_{23}W_{12} = W_{12}W_{13}W_{23}$.

- The operator $Q\colon e_{k,l} \mapsto q^{2(l-k)} e_{k,l}$ defined on $\operatorname{span}\{e_{i,j} \,|\, i,j \in \mathbb{Z}\}$ is closable and its closure Q is strictly positive and self-adjoint.
- Let $\overline{\ell_2(\mathbb{Z} \times \mathbb{Z})}$ denote the space of bounded linear functionals on $\ell_2(\mathbb{Z} \times \mathbb{Z})$. The antiunitary mapping

$$\ell_2(\mathbb{Z} \times \mathbb{Z}) \longrightarrow \overline{\ell_2(\mathbb{Z} \times \mathbb{Z})}$$

defined by $\xi \mapsto \langle \xi | \cdot \rangle$ will be denoted by $\xi \mapsto \overline{\xi}$.

THEOREM

The operator Q satisfies $W(Q \otimes Q)W^* = Q \otimes Q$. Furthermore there exists a unitary operator $\widetilde{W}$ on $\overline{\ell_2(\mathbb{Z} \times \mathbb{Z})} \otimes \ell_2(\mathbb{Z} \times \mathbb{Z})$ such that

$$\big\langle (\xi \otimes \theta) \big| W(\zeta \otimes \eta) \big\rangle = \Big\langle \big(\overline{\zeta} \otimes Q\theta\big) \Big| \widetilde{W}\big(\overline{\xi} \otimes Q^{-1}\eta\big) \Big\rangle$$

for all $\xi, \zeta \in \ell_2(\mathbb{Z} \times \mathbb{Z})$, $\theta \in \operatorname{Dom}(Q)$, and $\eta \in \operatorname{Dom}(Q^{-1})$

- The above results say that W is a **manageable multiplicative unitary** (a notion defined in 1997 by S.L. Woronowicz, building on the work of Baaj-Skandalis).
- Multiplicative unitaries are one of the keys to the theory of non-compact quantum groups and they provide a way to tell quantum groups from quantum semigroups.
- If G is a locally compact group and $\mathsf{L}_2(G)$ is the Hilbert space of functions on G square-integrable with respect to the right Haar measure then the associated multiplicative unitary W is the unitary operator on $\mathsf{L}_2(G)\otimes\mathsf{L}_2(G)\cong\mathsf{L}_2(G\times G)$ given by

$$(W\xi)(t,s)=\xi(t,ts),\qquad \xi\in\mathsf{L}_2(G\times G),\ t,s\in G.$$

- The quantum group described by $W=F_q(n^{-1}v\otimes vn)V$ from the previous slide is called $\mathrm{E}_q(2)$. It is a "quantum version" of the (double cover of) the group $\mathrm{E}(2)$ of rigid motions of the plane.

Quantum groups from multiplicative unitaries

- Let $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ be a manageable multiplicative unitary.
- Then $\mathsf{A} = \left\{(\omega \otimes \mathrm{id})W \,\middle|\, \omega \in \mathrm{B}(\mathcal{H})_*\right\}$ is a non-degenerate C*-subalgebra of $\mathrm{B}(\mathcal{H})$.
- $\mathsf{A} \ni a \mapsto W(a \otimes \mathbb{1})W^* \in \mathrm{M}(\mathsf{A} \otimes \mathsf{A})$ is a coassociative **morphism** $\mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$.
- The sets $\operatorname{span}\left\{\Delta(a)(\mathbb{1} \otimes b) \,\middle|\, a, b \in \mathsf{A}\right\}$ and $\operatorname{span}\left\{(a \otimes \mathbb{1})\Delta(b) \,\middle|\, a, b \in \mathsf{A}\right\}$ are dense in $\mathsf{A} \otimes \mathsf{A}$.
- There exists a (possibly unbounded) anti-multiplicative map $S\colon \mathsf{A} \to \mathsf{A}$ such that $S\big((\omega \otimes \mathrm{id})W\big) = (\omega \otimes \mathrm{id})W^*$ for all $\omega \in \mathrm{B}(\mathcal{H})_*$.
- $S = R \circ \tau_{\mathrm{i}/2}$, where R is a $*$-antiautomorphism of A such that $\Delta \circ R = (R \otimes R) \circ \Delta$ and $\{\tau_t\}_{t \in \mathbb{R}}$ is a one parameter group of automorphisms of A such that $\Delta \circ \tau_t = (\tau_t \otimes \tau_t) \circ \Delta$ for all t.

The hunt for the Haar measure

- The construction of a quantum group from a multiplicative unitary does not require the existence of the Haar measure.
- In all known examples the Haar measure can be constructed, yet we do not have a proof that it always exists.
- The Haar measure on a non-compact quantum group is a **weight** (not a state).
- For the quantum group $\mathrm{E}_q(2)$ the Haar measure was found by S. Baaj:
  - First we represent $\mathrm{C}_0(\mathrm{E}_2(2)) = \mathsf{B}_q$ on $\ell_2(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z})$ in such a way that v and n are the operators $e_{i,j,k} \mapsto e_{i,j+1,k+1}$, and $e_{i,j,k} \mapsto q^j e_{i+1,j,k}$.
  - For a positive $a \in \mathrm{C}_0(\mathrm{E}_q(2))$ considered as an operator on $\ell_2(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z})$ we put

$$\boldsymbol{h}(a) = \sum_{j\in\mathbb{Z}} q^{2j} \big\langle e_{0,j,0} \big| a e_{0,j,0} \big\rangle.$$

  - Then $\boldsymbol{h}$ is the right and left invariant Haar mesure ($\mathrm{E}_q(2)$ is unimodular).

Things I chose not to talk about

- Differential calculi on quantum groups,
- Pontriagin duality, Fourier transform,
- quantum subgroups (closed, open, integrable, etc.),
- homomorphisms of quantum groups,
- actions of quantum groups, homogeneous spaces, etc.,
- quantum symmetries of non-commutative objects,
- Bohr compactifications, property (T), amenability, etc.,
- applications to non-local games, quantum isomorphisms of (quantum) graphs,
- quantum groups on von Neumann algebra level,
- and many others.

SOME BIBLIOGRAPHY

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Thank you for your attention