QUANTUM CORRELATIONS ON QUANTUM SPACES

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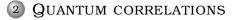
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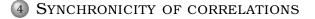
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QUANTUM CORRELATIONS









WHAT ARE QUANTUM SPACES?

DEFINITION

A **quantum space** is an object of the category dual to the category of C^* -algebras.

This means:

- There is a bijection between C*-algebras and quantum spaces.
- Mappings of quantum spaces are morphisms between C*-algebras (in the opposite direction).

 $\begin{array}{c} X \longleftrightarrow \mathrm{C}(X) \\ \mathbb{X} \longleftrightarrow \mathrm{C}(\mathbb{X}) \\ \Phi \colon \mathrm{C}(\mathbb{O}) \longrightarrow \mathrm{C}(\mathbb{P}) \\ \text{is regarded as a map} \\ \mathbb{P} \longrightarrow \mathbb{O} \end{array}$

Despite its merely linguistic nature, the notion of a quantum space can lead to interesting mathematical investigations.

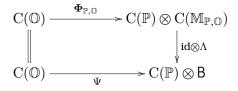
- Let \mathbb{P} and \mathbb{O} be quantum spaces.
- Sometimes there exits a quantum space $\mathbb{M}_{\mathbb{P},\mathbb{O}}$ equipped with

 $\Phi_{\mathbb{P},\mathbb{O}}\colon \operatorname{C}(\mathbb{O}) \longrightarrow \operatorname{C}(\mathbb{P}) \otimes \operatorname{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$

(a quantum family of maps $\mathbb{P} \to \mathbb{O}$ indexed by $\mathbb{M}_{\mathbb{P},\mathbb{O}}$) s.t.

- for any C*-algebra B
- for any $\Psi \colon \operatorname{C}(\mathbb{O}) \to \operatorname{C}(\mathbb{P}) \otimes \mathsf{B}$

there exists a unique $\Lambda \colon \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \to \mathsf{B}$ such that



• $M_{\mathbb{P},\mathbb{O}}$ is the quantum space of maps $\mathbb{P} \to \mathbb{O}$.

- A quantum space ℙ is **finite** if the corresponding C*-algebra C(ℙ) is finite-dimensional.
- Let \mathbb{P} and \mathbb{O} be finite quantum spaces.
- A **quantum correlation** with quantum question set P and quantum answer set O is a u.c.p. map

 $T\colon \operatorname{C}(\mathbb{O})\otimes\operatorname{C}(\mathbb{O})\longrightarrow\operatorname{C}(\mathbb{P})\otimes\operatorname{C}(\mathbb{P}).$

• A correlation *T* as above is **non-signalling** if

 $T(\mathcal{C}(\mathbb{O})\otimes\mathbb{1})\subset\mathcal{C}(\mathbb{P})\otimes\mathbb{1}$ and $T(\mathbb{1}\otimes\mathcal{C}(\mathbb{O}))\subset\mathbb{1}\otimes\mathcal{C}(\mathbb{P}).$

- A POVM on a finite quantum space \mathbb{O} is a u.c.p. map $C(\mathbb{O}) \to B(H)$.
- A **quantum family** of POVMs on \mathbb{O} indexed by \mathbb{P} is a u.c.p. map $C(\mathbb{O}) \to C(\mathbb{P}) \otimes B(H)$.
- Given two such families $R, S: C(\mathbb{O}) \to C(\mathbb{P}) \otimes B(\mathsf{H})$ satisfying

 $\forall x, y \in C(\mathbb{O}) \quad R(x)_{13}S(y)_{23} = S(y)_{23}R(x)_{13}$

and a state ω on B(H) the map $T: C(\mathbb{O}) \otimes C(\mathbb{O}) \to C(\mathbb{P}) \otimes C(\mathbb{P})$ defined by

$$T(\mathbf{x} \otimes \mathbf{y}) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega) \big(R(\mathbf{x})_{13} S(\mathbf{y})_{23} \big)$$

is is a non-signalling correlation.

- This is certainly not the most general construction.
- We call correlations *T* as constructed above the **realizable** ones.

- Let $R: C(\mathbb{O}) \to C(\mathbb{P}) \otimes B(\mathsf{H})$ be a u.c.p. map.
- Then the span S of the slices

$$\{(\phi \otimes \mathrm{id})R(x) \mid x \in \mathrm{C}(\mathbb{O}), \phi \in \mathrm{C}(\mathbb{P})^*\}$$

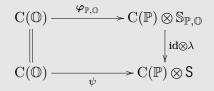
is an operator system in B(H).

• In particular, construction of realizable non-signalling correlations involves u.c.p. maps $C(\mathbb{O}) \to C(\mathbb{P}) \otimes S$, where S is an operator system.

THEOREM

Let ${\mathbb P}$ and ${\mathbb O}$ be finite quantum spaces. Then

ⓐ there exists an operator system S_{P,0} and a u.c.p. map $\varphi_{\mathbb{P},0}$: C(0) → C(P) ⊗ S_{P,0} such that for any operator system S and any u.c.p. map ψ : C(0) → C(P) ⊗ S there exists a unique u.c.p. map λ : S_{P,0} → S such that the diagram



commutes,

2) the C^{*}-envelope of $S_{\mathbb{P},\mathbb{O}}$ is $C(\mathbb{M}_{\mathbb{P},\mathbb{O}})$.

• In fact we prove that $S_{\mathbb{P},\mathbb{O}} \subset C(M_{\mathbb{P},\mathbb{O}})$ is **hyperrigid**.

THEOREM

Let $T: C(\mathbb{O}) \otimes C(\mathbb{O}) \to C(\mathbb{P}) \otimes C(\mathbb{P})$ be a quantum non-signalling correlation. Then the following are equivalent:

- **③** $T(x \otimes y) = (id \otimes id \otimes \omega_{\xi}) (R(x)_{13}S(y)_{23})$ for some u.c.p. maps *R*, S: C(ℂ) → C(ℙ) ⊗ B(H) with commuting right legs and a unit vector $\xi \in H$,
- ② $T(x \otimes y) = (id \otimes id \otimes \omega_{\xi}) (R(x)_{13}S(y)_{23})$ for *-homomorphisms *R*, *S*: C(ℂ) → C(ℙ) ⊗ B(H) with commuting right legs and a unit vector $\xi \in H$,
- $\begin{tabular}{ll} \Im & T(x\otimes y) = (\mathrm{id}\otimes\mathrm{id}\otimes s) \big(\Phi_{\mathbb{P},\mathbb{O}}(x)_{13} \Phi_{\mathbb{P},\mathbb{O}}(y)_{24} \big) \mbox{ for a state s on $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})\otimes_{\max}\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$, } \end{tabular}$
- $\begin{array}{l} \textcircled{4} \quad T(x\otimes y) = (\operatorname{id}\otimes\operatorname{id}\otimes s) \big(\varphi_{\mathbb{P},\mathbb{O}}(x)_{13}\varphi_{\mathbb{P},\mathbb{O}}(y)_{24} \big) \text{ for a state s on } \\ \mathbb{S}_{\mathbb{P},\mathbb{O}}\otimes_{\mathrm{c}} \mathbb{S}_{\mathbb{P},\mathbb{O}}. \end{array}$

- The assignment $(\mathbb{P}, \mathbb{O}) \mapsto \mathbb{M}_{\mathbb{P}, \mathbb{O}}$ is a bi-functor.
- With \mathbb{P} fixed it is the left adjoint functor to $C(\mathbb{P}) \otimes _$.

THEOREM

 $\mathbb{P}, \mathbb{P}_1, \mathbb{P}_2$ — finite quantum spaces, $\mathbb{O}, \mathbb{O}_1, \mathbb{O}_2$ — quantum spaces with $C(\mathbb{O}), C(\mathbb{O}_1), C(\mathbb{O}_2)$ unital and finitely generated. Then

- $\begin{array}{l} \textcircled{0} \quad any \ \pi \colon \operatorname{C}(\mathbb{O}_2) \to \operatorname{C}(\mathbb{O}_1) \ gives \ rise \ to \ a \ unique \\ \Lambda \colon \operatorname{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}_2}) \to \operatorname{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}_1}) \ s.t. \ \Phi_{\mathbb{P},\mathbb{O}_1} \circ \pi = (\operatorname{id} \otimes \Lambda) \circ \Phi_{\mathbb{P},\mathbb{O}_2}, \end{array}$
- $\begin{array}{l} @ \text{ any } \rho \colon \mathrm{C}(\mathbb{P}_1) \to \mathrm{C}(\mathbb{P}_2) \text{ gives rise to a unique} \\ \widetilde{\Lambda} \colon \mathrm{C}(\mathbb{M}_{\mathbb{P}_2,\mathbb{O}}) \to \mathrm{C}(\mathbb{M}_{\mathbb{P}_1,\mathbb{O}}) \text{ s.t. } (\rho \otimes \mathrm{id}) \circ \Phi_{\mathbb{P}_1,\mathbb{O}} = (\mathrm{id} \otimes \widetilde{\Lambda}) \circ \Phi_{\mathbb{P}_2,\mathbb{O}}, \end{array}$
- 3) if π is surjective then so is Λ ,
- (4) if ρ is injective then $\widetilde{\Lambda}$ is surjective,
- 5 if π is injective then so is Λ ,
- **(6)** if ρ is surjective then $\widetilde{\Lambda}$ is injective.

• A unital C*-algebra A has the **lifting property** if for any unital C*-algebra B with an ideal $J \subset B$ and any u.c.p. map $\varphi \colon A \to B/J$ there exists a u.c.p. map $\widetilde{\varphi} \colon A \to B$ such that $\varphi = q \circ \widetilde{\varphi}$, where q is the quotient map $B \to B/J$.

THEOREM

Let \mathbb{P} and \mathbb{O} be finite quantum spaces. Then the C^{*}-algebra $C(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ has the lifting property.

- The key elements of the proof are similar to those introduced by Brannan-Ganesan-Harris in the case of classical \mathbb{O} .
- A major role is played by Kasparov's dilation theorem.

Remark

 $C(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ is almost never nuclear (dim $C(\mathbb{O})$, dim $C(\mathbb{P}) > 1$ and dim $C(\mathbb{O}) \cdot \dim C(\mathbb{P}) > 4$). • A C*-algebra A is **residually finite-dimensional** if A has a separating family of finite-dimensional representations.

THEOREM

Let \mathbb{P} and \mathbb{O} be finite quantum spaces. Then the C^{*}-algebra $C(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ is residually finite-dimensional.

- Again we generalize some ideas of Brannan-Ganesan-Harris.
- Functorial properties are helpful in the proof.
- RFD C*-algebras have many traces.

Let

$$\mathrm{C}(\mathbb{P}) = \bigoplus_{l=1}^{N_{\mathbb{P}}} \mathsf{Mat}_{m_l}(\mathbb{C}), \quad \mathrm{C}(\mathbb{O}) = \bigoplus_{k=1}^{N_{\mathbb{O}}} \mathsf{Mat}_{n_k}(\mathbb{C}).$$

• Let $T: C(\mathbb{O}) \otimes C(\mathbb{O}) \to C(\mathbb{P}) \otimes C(\mathbb{P})$ be a correlation.

• With bases of matrix units

$$\left\{ \begin{array}{l} {}^{l}f_{st} \left| \ l = 1, \dots, N_{\mathbb{P}}, \ s, t = 1, \dots, m_{l} \right\} \\ \left\{ \begin{array}{l} {}^{k}e_{ij} \left| \ k = 1, \dots, N_{\mathbb{O}}, \ i, j = 1, \dots, n_{k} \right\} \end{array} \right.$$

we associate the matrix of *T*:

$$T({}^{\mathbf{k}}e_{\mathbf{jj}} \otimes {}^{\mathbf{k'}}e_{\mathbf{i'j'}}) = \sum_{\mathbf{s},\mathbf{s'},t,t',l,l'} {}^{\mathbf{kk'}}_{ll'} X^{(\mathbf{st})(\mathbf{s't'})}_{(\mathbf{ij})(\mathbf{i'j'})} {}^{l}f_{\mathbf{st}} \otimes {}^{l'}f_{\mathbf{s't'}}$$

• *T* is **synchronous** if

$$\sum_{s,t,i,j,k,l} \frac{1}{n_k m_l} {}^{kk}_{ll} X^{(st)(st)}_{(ij)(ij)} = N_{\mathbb{P}}.$$

PROPOSITION

Let

$$\phi = \frac{1}{\sqrt{N_{\mathbb{P}}}} \sum_{l=1}^{N_{\mathbb{P}}} \frac{1}{\sqrt{m_l}} \sum_{s=1}^{m_l} {}^l f_s \otimes {}^l f_s \in \left(\bigoplus_{l=1}^{N_{\mathbb{P}}} \mathbb{C}^{m_l} \right) \otimes \left(\bigoplus_{l=1}^{N_{\mathbb{P}}} \mathbb{C}^{m_l} \right)$$

Then T is synchronous if and only if

$$\left\langle \phi \left| T\left(\sum_{i,j,k} \frac{1}{n_k} {}^k e_{ij} \otimes {}^k e_{ij}\right) \phi \right\rangle = 1.$$

- Let τ be a trace on $C(\mathbb{M}_{\mathbb{P},\mathbb{O}})$.
- Then τ defines a trace $\hat{\tau}$ on $C(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \otimes_{\max} C(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ (this involves $C(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \cong C(\mathbb{M}_{\mathbb{P},\mathbb{O}})^{\text{op}}$).
- $\hat{\tau}$ defines a realizable non-signalling correlation T_{τ} .

Fact

 T_{τ} is synchronous.

THEOREM

Let *T* be a synchronous realizable non-signalling correlation. Then $T = T_{\tau}$ for a certain trace τ on $C(\mathbb{M}_{\mathbb{P},\mathbb{O}})$.

Thank you!