Quantum correlations on quantum spaces

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What are quantum spaces?

Definition

A **quantum space** is an object of the category dual to the category of C*-algebras.

This means:

- There is a bijection between C*-algebras and quantum spaces.

  $X \longleftrightarrow \mathrm{C}(X)$

  $\mathbb{X} \longleftrightarrow \mathrm{C}(\mathbb{X})$

- Mappings of quantum spaces are morphisms between C*-algebras (in the opposite direction).

  $\Phi\colon\ \mathrm{C}(\mathbb{O}) \longrightarrow \mathrm{C}(\mathbb{P})$

  is regarded as a map

  $\mathbb{P} \longrightarrow \mathbb{O}$

Despite its merely linguistic nature, the notion of a quantum space can lead to interesting mathematical investigations.

- Let $\mathbb{P}$ and $\mathbb{O}$ be quantum spaces.
- Sometimes there exits a quantum space $\mathbb{M}_{\mathbb{P},\mathbb{O}}$ equipped with

$$\boldsymbol{\Phi}_{\mathbb{P},\mathbb{O}}\colon\ \mathrm{C}(\mathbb{O})\longrightarrow\mathrm{C}(\mathbb{P})\otimes\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$$

(a **quantum family of maps** $\mathbb{P}\to\mathbb{O}$ indexed by $\mathbb{M}_{\mathbb{P},\mathbb{O}}$) s.t.
  - for any C*-algebra B
  - for any $\Psi\colon\ \mathrm{C}(\mathbb{O})\to\mathrm{C}(\mathbb{P})\otimes\mathsf{B}$

there exists a unique $\Lambda\colon\ \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})\to\mathsf{B}$ such that

$$\begin{array}{ccc}\mathrm{C}(\mathbb{O}) & \xrightarrow{\ \boldsymbol{\Phi}_{\mathbb{P},\mathbb{O}}\ } & \mathrm{C}(\mathbb{P})\otimes\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})\\ \Big\| & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\Lambda}\\ \mathrm{C}(\mathbb{O}) & \xrightarrow[\ \Psi\ ]{} & \mathrm{C}(\mathbb{P})\otimes\mathsf{B}\end{array}$$

- $\mathbb{M}_{\mathbb{P},\mathbb{O}}$ is the **quantum space of maps** $\mathbb{P}\to\mathbb{O}$.

- A quantum space $\mathbb{P}$ is **finite** if the corresponding C*-algebra $\mathrm{C}(\mathbb{P})$ is finite-dimensional.
- Let $\mathbb{P}$ and $\mathbb{O}$ be finite quantum spaces.
- A **quantum correlation** with quantum question set $\mathbb{P}$ and quantum answer set $\mathbb{O}$ is a u.c.p. map

$$T\colon \mathrm{C}(\mathbb{O})\otimes\mathrm{C}(\mathbb{O})\longrightarrow\mathrm{C}(\mathbb{P})\otimes\mathrm{C}(\mathbb{P}).$$

- A correlation T as above is **non-signalling** if

$$T\bigl(\mathrm{C}(\mathbb{O})\otimes\mathbb{1}\bigr)\subset\mathrm{C}(\mathbb{P})\otimes\mathbb{1}\quad\text{and}\quad T\bigl(\mathbb{1}\otimes\mathrm{C}(\mathbb{O})\bigr)\subset\mathbb{1}\otimes\mathrm{C}(\mathbb{P}).$$

- A POVM on a finite quantum space $\mathbb{O}$ is a u.c.p. map $\mathrm{C}(\mathbb{O}) \to \mathrm{B}(\mathsf{H})$.
- A **quantum family** of POVMs on $\mathbb{O}$ indexed by $\mathbb{P}$ is a u.c.p. map $\mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathrm{B}(\mathsf{H})$.
- Given two such families $R, S\colon \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathrm{B}(\mathsf{H})$ satisfying

$$\forall\, x, y \in \mathrm{C}(\mathbb{O}) \quad R(x)_{13}S(y)_{23} = S(y)_{23}R(x)_{13}$$

  and a state ω on $\mathrm{B}(\mathsf{H})$ the map $T\colon \mathrm{C}(\mathbb{O}) \otimes \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathrm{C}(\mathbb{P})$ defined by

$$T(x \otimes y) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega)\big(R(x)_{13}S(y)_{23}\big)$$

  is is a non-signalling correlation.
- This is certainly not the most general construction.
- We call correlations T as constructed above the **realizable** ones.

- Let $R\colon \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathrm{B}(\mathsf{H})$ be a u.c.p. map.
- Then the span S of the slices

$$\left\{(\phi \otimes \mathbf{id})R(x) \,\middle|\, x \in \mathrm{C}(\mathbb{O}),\ \phi \in \mathrm{C}(\mathbb{P})^*\right\}$$

  is an operator system in $\mathrm{B}(\mathsf{H})$.
- In particular, construction of realizable non-signalling correlations involves u.c.p. maps $\mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes S$, where S is an operator system.

Theorem

Let $\mathbb{P}$ and $\mathbb{O}$ be finite quantum spaces. Then

1. *there exists an operator system $\mathbb{S}_{\mathbb{P},\mathbb{O}}$ and a u.c.p. map $\boldsymbol{\varphi}_{\mathbb{P},\mathbb{O}}\colon \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathbb{S}_{\mathbb{P},\mathbb{O}}$ such that for any operator system S and any u.c.p. map $\psi\colon \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathsf{S}$ there exists a unique u.c.p. map $\lambda\colon \mathbb{S}_{\mathbb{P},\mathbb{O}} \to \mathsf{S}$ such that the diagram*

$$\begin{array}{ccc} \mathrm{C}(\mathbb{O}) & \xrightarrow{\ \boldsymbol{\varphi}_{\mathbb{P},\mathbb{O}}\ } & \mathrm{C}(\mathbb{P}) \otimes \mathbb{S}_{\mathbb{P},\mathbb{O}} \\ \Big\| & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\lambda} \\ \mathrm{C}(\mathbb{O}) & \xrightarrow[\ \psi\ ]{} & \mathrm{C}(\mathbb{P}) \otimes \mathsf{S} \end{array}$$

   commutes,

2. *the C^*-envelope of $\mathbb{S}_{\mathbb{P},\mathbb{O}}$ is $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$.*

- In fact we prove that $\mathbb{S}_{\mathbb{P},\mathbb{O}} \subset \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ is **hyperrigid**.

Theorem

Let $T\colon \mathrm{C}(\mathbb{O})\otimes\mathrm{C}(\mathbb{O})\to\mathrm{C}(\mathbb{P})\otimes\mathrm{C}(\mathbb{P})$ *be a quantum non-signalling correlation. Then the following are equivalent:*

1. $T(x\otimes y)=(\mathrm{id}\otimes\mathrm{id}\otimes\omega_\xi)\bigl(R(x)_{13}S(y)_{23}\bigr)$ *for some u.c.p. maps* $R,S\colon \mathrm{C}(\mathbb{O})\to\mathrm{C}(\mathbb{P})\otimes\mathrm{B}(\mathsf{H})$ *with commuting right legs and a unit vector* $\xi\in\mathsf{H}$,
2. $T(x\otimes y)=(\mathrm{id}\otimes\mathrm{id}\otimes\omega_\xi)\bigl(R(x)_{13}S(y)_{23}\bigr)$ *for* $*$*-homomorphisms* $R,S\colon \mathrm{C}(\mathbb{O})\to\mathrm{C}(\mathbb{P})\otimes\mathrm{B}(\mathsf{H})$ *with commuting right legs and a unit vector* $\xi\in\mathsf{H}$,
3. $T(x\otimes y)=(\mathrm{id}\otimes\mathrm{id}\otimes s)\bigl(\boldsymbol{\Phi}_{\mathbb{P},\mathbb{O}}(x)_{13}\boldsymbol{\Phi}_{\mathbb{P},\mathbb{O}}(y)_{24}\bigr)$ *for a state* s *on* $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})\otimes_{\mathrm{max}}\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$,
4. $T(x\otimes y)=(\mathrm{id}\otimes\mathrm{id}\otimes s)\bigl(\varphi_{\mathbb{P},\mathbb{O}}(x)_{13}\varphi_{\mathbb{P},\mathbb{O}}(y)_{24}\bigr)$ *for a state* s *on* $\mathbb{S}_{\mathbb{P},\mathbb{O}}\otimes_{\mathrm{c}}\mathbb{S}_{\mathbb{P},\mathbb{O}}$.

- The assignment $(\mathbb{P},\mathbb{O}) \longmapsto \mathbb{M}_{\mathbb{P},\mathbb{O}}$ is a bi-functor.
- With $\mathbb{P}$ fixed it is the left adjoint functor to $\mathrm{C}(\mathbb{P}) \otimes \_\_$.

THEOREM

$\mathbb{P}, \mathbb{P}_1, \mathbb{P}_2$ — *finite quantum spaces,* $\mathbb{O}, \mathbb{O}_1, \mathbb{O}_2$ — *quantum spaces with* $\mathrm{C}(\mathbb{O}), \mathrm{C}(\mathbb{O}_1), \mathrm{C}(\mathbb{O}_2)$ *unital and finitely generated. Then*

1. *any* $\pi\colon \mathrm{C}(\mathbb{O}_2) \to \mathrm{C}(\mathbb{O}_1)$ *gives rise to a unique* $\Lambda\colon \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}_2}) \to \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}_1})$ *s.t.* $\boldsymbol{\Phi}_{\mathbb{P},\mathbb{O}_1} \circ \pi = (\mathrm{id} \otimes \Lambda) \circ \boldsymbol{\Phi}_{\mathbb{P},\mathbb{O}_2}$,
2. *any* $\rho\colon \mathrm{C}(\mathbb{P}_1) \to \mathrm{C}(\mathbb{P}_2)$ *gives rise to a unique* $\widetilde{\Lambda}\colon \mathrm{C}(\mathbb{M}_{\mathbb{P}_2,\mathbb{O}}) \to \mathrm{C}(\mathbb{M}_{\mathbb{P}_1,\mathbb{O}})$ *s.t.* $(\rho \otimes \mathrm{id}) \circ \boldsymbol{\Phi}_{\mathbb{P}_1,\mathbb{O}} = (\mathrm{id} \otimes \widetilde{\Lambda}) \circ \boldsymbol{\Phi}_{\mathbb{P}_2,\mathbb{O}}$,
3. *if* π *is surjective then so is* Λ,
4. *if* ρ *is injective then* $\widetilde{\Lambda}$ *is surjective,*
5. *if* π *is injective then so is* Λ,
6. *if* ρ *is surjective then* $\widetilde{\Lambda}$ *is injective.*

- A unital C*-algebra A has the **lifting property** if for any unital C*-algebra B with an ideal $\mathsf{J} \subset \mathsf{B}$ and any u.c.p. map $\varphi\colon \mathsf{A} \to \mathsf{B}/\mathsf{J}$ there exists a u.c.p. map $\tilde{\varphi}\colon \mathsf{A} \to \mathsf{B}$ such that $\varphi = q \circ \tilde{\varphi}$, where q is the quotient map $\mathsf{B} \to \mathsf{B}/\mathsf{J}$.

THEOREM

Let $\mathbb{P}$ and $\mathbb{O}$ be finite quantum spaces. Then the C^**-algebra* $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ *has the lifting property.*

- The key elements of the proof are similar to those introduced by Brannan-Ganesan-Harris in the case of classical $\mathbb{O}$.
- A major role is played by Kasparov's dilation theorem.

REMARK

$\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ is almost never nuclear
($\dim \mathrm{C}(\mathbb{O}), \dim \mathrm{C}(\mathbb{P}) > 1$ and $\dim \mathrm{C}(\mathbb{O}) \cdot \dim \mathrm{C}(\mathbb{P}) > 4$).

- A C*-algebra A is **residually finite-dimensional** if A has a separating family of finite-dimensional representations.

THEOREM

Let $\mathbb{P}$ and $\mathbb{O}$ be finite quantum spaces. Then the C-algebra $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ is residually finite-dimensional.*

- Again we generalize some ideas of Brannan-Ganesan-Harris.
- Functorial properties are helpful in the proof.
- RFD C*-algebras have many traces.

- Let

$$\mathrm{C}(\mathbb{P})=\bigoplus_{l=1}^{N_{\mathbb{P}}}\mathsf{Mat}_{m_l}(\mathbb{C}),\quad \mathrm{C}(\mathbb{O})=\bigoplus_{k=1}^{N_{\mathbb{O}}}\mathsf{Mat}_{n_k}(\mathbb{C}).$$

- Let $T\colon\ \mathrm{C}(\mathbb{O})\otimes\mathrm{C}(\mathbb{O})\to\mathrm{C}(\mathbb{P})\otimes\mathrm{C}(\mathbb{P})$ be a correlation.
- With bases of matrix units

$$\bigl\{\,{}^{l}f_{st}\,\big|\,l=1,\dots,N_{\mathbb{P}},\ s,t=1,\dots,m_l\bigr\}$$
$$\bigl\{\,{}^{k}e_{ij}\,\big|\,k=1,\dots,N_{\mathbb{O}},\ i,j=1,\dots,n_k\bigr\}$$

we associate the matrix of T:

$$T({}^{k}e_{ij}\otimes{}^{k'}e_{i'j'})=\sum_{s,s',t,t',l,l'}{}^{kk'}_{ll'}X^{(st)(s't')}_{(ij)(i'j')}\,{}^{l}f_{st}\otimes{}^{l'}f_{s't'}$$

- T is **synchronous** if

$$\sum_{s,t,i,j,k,l}\frac{1}{n_km_l}\,{}^{kk}_{ll}X^{(st)(st)}_{(ij)(ij)}=N_{\mathbb{P}}.$$

Proposition

Let

$$\phi = \frac{1}{\sqrt{N_{\mathbb{P}}}} \sum_{l=1}^{N_{\mathbb{P}}} \frac{1}{\sqrt{m_l}} \sum_{s=1}^{m_l} {}^{l}f_s \otimes {}^{l}f_s \in \left(\bigoplus_{l=1}^{N_{\mathbb{P}}} \mathbb{C}^{m_l} \right) \otimes \left(\bigoplus_{l=1}^{N_{\mathbb{P}}} \mathbb{C}^{m_l} \right)$$

Then T is synchronous if and only if

$$\left\langle \phi \middle| T\left(\sum_{i,j,k} \frac{1}{n_k} {}^{k}e_{ij} \otimes {}^{k}e_{ij} \right) \phi \right\rangle = 1.$$

- Let τ be a trace on $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$.
- Then τ defines a trace $\hat{\tau}$ on $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \otimes_{\max} \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ (this involves $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \cong \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})^{\mathrm{op}}$).
- $\hat{\tau}$ defines a realizable non-signalling correlation T_τ.

Fact

T_τ is synchronous.

Theorem

Let T be a synchronous realizable non-signalling correlation. Then $T = T_\tau$ for a certain trace τ on $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$.

Thank you!