On the characterization of quantum $\mathrm{SO}(3)$ groups

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Operator algebraic aspects of quantum groups

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Outline of talk

- $q \in]0, 1]$
- $\mathrm{C}(\mathrm{S}_q\mathrm{U}(2))$ — C^*-algebra of functions on $\mathrm{S}_q\mathrm{U}(2)$
- α, γ — standard generators of $\mathrm{C}(\mathrm{S}_q\mathrm{U}(2))$

Definition

$\mathrm{C}(\mathrm{S}_q\mathrm{O}(3))$ is the subalgebra of $\mathrm{C}(\mathrm{S}_q\mathrm{U}(2))$ generated by matrix elements of the spin-1 representation:

$$\begin{bmatrix} \alpha^{*2} & -(q^2+1)\alpha^*\gamma & -q\gamma^2 \\ \gamma^*\alpha^* & \mathbb{1}-(q^2+1)\gamma^*\gamma & \alpha\gamma \\ -q\gamma^{*2} & -(q^2+1)\gamma^*\alpha & \alpha^2 \end{bmatrix}$$

$\Delta_{\mathrm{S}_q\mathrm{O}(3)} = \Delta_{\mathrm{S}_q\mathrm{U}(2)}\big|_{\mathrm{C}(\mathrm{S}_q\mathrm{O}(3))}$.

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Theorem (Podleś)

$\mathrm{C}(\mathrm{S}_q\mathrm{O}(3))$ is the universal C^*-algebra generated by A, C, G, K, L such that

$$\begin{aligned} L^*L &= (\mathbb{1}-K)(\mathbb{1}-q^{-2}K), & AK &= q^2KA, \\ LL^* &= (\mathbb{1}-q^2K)(\mathbb{1}-q^4K), & CK &= q^2KC, \\ G^*G &= GG^*, & LG &= q^4GL, \\ K^2 &= G^*G, & LA &= q^2AL, \\ A^*A &= K-K^2, & AG &= q^2GA, \\ AA^* &= q^2K-q^4K^2, & AC &= CA, \\ C^*C &= K-K^2, & LG^* &= q^4G^*L, \\ CC^* &= q^2K-q^4K^2, & A^2 &= q^{-1}LG, \\ LK &= q^4KL, & A^*L &= q^{-1}(\mathbb{1}-K)C, \\ GK &= KG, & K^* &= K. \end{aligned}$$

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Lemma

Let $\mathcal{H}$ be a Hilbert space and $A, C, K \in \mathrm{B}(\mathcal{H})$ with $0 \leq K \leq \mathbb{1}$. Fix $q \in [0, 1[$. Assume

$$A^*A = C^*C = K - K^2,$$
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Then

$$CK = q^2KC$$

- The lemma is not true for $q = 1$.
- $[A, C] = 0$ is not necessary for most q, but there are exceptional values.

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Let **A** be the universal C^*-algebra generated by β,γ,δ such that

$$\begin{aligned}
q^4\delta^*\delta+\gamma^*\gamma+q^4\delta\delta^*+\beta\beta^*&=\mathbb{1}, & \beta\gamma&=-q^4\delta^2,\\
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q^4\delta\delta^*+\beta\beta^*+q^2\gamma\gamma^*+q^2\delta\delta^*&=\mathbb{1}, & \delta\gamma&=q^2\gamma\delta,\\
\gamma^*\delta-q^2\delta^*\beta+\beta\delta^*-q^2\delta\gamma^*&=0.
\end{aligned}$$

A has comultiplication $\Delta_{\mathbf{A}}\in\mathrm{Mor}(\mathbf{A},\mathbf{A}\otimes\mathbf{A})$:

$$\begin{aligned}
\Delta_{\mathbf{A}}(\beta)&=q^4\delta\gamma^*\otimes\delta-q^2\beta\delta^*\otimes\delta+\beta\otimes\beta+\gamma^*\otimes\gamma-q^2\delta^*\beta\otimes\delta+\gamma^*\delta\otimes\delta,\\
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- $(\mathbf{A}, \Delta_{\mathbf{A}})$ acts on M_2:

$$\mathbf{\Phi} : M_2 \ni \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} -q^2\delta & \beta \\ \gamma & \delta \end{bmatrix} \in M_2(\mathbf{A}) = M_2 \otimes \mathbf{A}$$

- This action preserves the Powers state:

$$(\omega_q \otimes \mathrm{id})\mathbf{\Phi}(m) = \omega_q(m)\mathbb{1}$$

for all $m \in M_2$, where

$$\omega_q\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{1+q^2}(a + q^2 d)$$

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Proposition

For any C^**-algebra* D *and any* $\Psi \in \mathrm{Mor}(M_2, M_2 \otimes D)$ *such that*

$$(\omega_q \otimes \mathrm{id})\Psi(m) = \omega_q(m)\mathbb{1}$$

for all $m \in M_2$, *there exists a unique* $\Lambda \in \mathrm{Mor}(\mathbf{A}, D)$ *such that*

$$\begin{array}{ccc} M_2 & \xrightarrow{\quad\Phi\quad} & M_2 \otimes \mathbf{A} \\ \Big\| & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\Lambda} \\ M_2 & \xrightarrow{\quad\Psi\quad} & M_2 \otimes D \end{array}$$

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- $\mathrm{S}_q\mathrm{O}(3)$ acts on M_2:

$$\Psi_q : M_2 \ni \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} -qA & L \\ -qG & q^{-1}A \end{bmatrix} \in M_2 \otimes \mathrm{C}\big(\mathrm{S}_q\mathrm{O}(3)\big)$$

- This action preserves the Powers state
- Thus there is $\Lambda_q \in \mathrm{Mor}\big(\mathbf{A}, \mathrm{C}(\mathrm{S}_q\mathrm{O}(3))\big)$ with

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Let $\Psi_{\mathcal{G}}(M_2)(\mathbb{1}\otimes B)=M_2\otimes B$

- $\mathcal{G} = (B, \Delta_B)$ be a compact quantum group
- $\Psi_{\mathcal{G}} \in \mathrm{Mor}(M_2, M_2 \otimes B)$ be a continuous action of $\mathcal{G}$
- Assume that $\Psi_{\mathcal{G}}$ preserves ω_q
- We know: there is a unique $\Lambda \in \mathrm{Mor}(\mathbf{A}, B)$ with

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There exists a unique $\Gamma \in \mathrm{Mor}\big(\mathrm{C}(\mathrm{S}_q\mathrm{O}(3)), B\big)$ *such that*

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- $\mathcal{G} = (B, \Delta_B)$ be a compact quantum group
- $\Psi_{\mathcal{G}} \in \mathrm{Mor}(M_2, M_2 \otimes B)$ be a continuous action of $\mathcal{G}$
- Assume that $\Psi_{\mathcal{G}}$ preserves ω_q
- We know: there is a unique $\Lambda \in \mathrm{Mor}(\mathbf{A}, B)$ with

$$(\mathrm{id} \otimes \Lambda)\circ\mathbf{\Phi} = \Psi_{\mathcal{G}}$$

Theorem

There exists a unique $\Gamma \in \mathrm{Mor}\big(\mathrm{C}(\mathrm{S}_q\mathrm{O}(3)), B\big)$ *such that*

$$\Lambda = \Gamma\circ\Lambda_q$$

Moreover

- Γ intertwines the actions of $\mathcal{G}$ and $\mathrm{S}_q\mathrm{O}(3)$:

$$(\mathrm{id}\otimes\Gamma)\circ\Psi_q=\Psi_{\mathcal{G}}$$

  (and this condition determines Γ uniquely)
- Γ is a compact quantum group morphism
- Cases of $q=0$ and $q=1$ also are completely understood

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Theorem

Let $\mathcal{S} = (D, \Delta_D)$ *be a quantum semigroup acting continuously on* M_2 *with* $\Psi_{\mathcal{S}} \in \mathrm{Mor}(M_2, M_2 \otimes D)$ *and preserving a faithful state* ω*.*
Then there exists a $q \in]0,1]$*,* $u \in M_2$ *— unitary and* $\Gamma \in \mathrm{Mor}\big(\mathrm{C}(\mathrm{S}_q\mathrm{O}(3)), D\big)$ *such that for each* $m \in M_2$

$$\Psi_{\mathcal{S}}(m) = (\mathrm{id} \otimes \Gamma)\big((u \otimes \mathbb{1})\Psi_q(u^* m u)(u^* \otimes \mathbb{1})\big).$$

Moreover u *and* Γ *are unique for each* q *and* q *depends on* ω*.*

- Any such action factors through the $\mathfrak{b}\mathcal{S}$ (a *compact q. group*)
- Any faithful state on M_2 is conjugate to a unique Powers state
- Ergodic actions have unique invariant states
- Any continuous compact q.g. action has a faithful inv. state

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Let $\mathcal{S} = (D, \Delta_D)$ be a quantum semigroup acting continuously on M_2 with $\Psi_\mathcal{S} \in \mathrm{Mor}(M_2, M_2 \otimes D)$ and preserving a faithful state ω. Then there exists a $q \in]0,1]$, $u \in M_2$ — unitary and $\Gamma \in \mathrm{Mor}\big(\mathrm{C}(\mathrm{S}_q\mathrm{O}(3)), D\big)$ such that for each $m \in M_2$

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