Compact Quantum Metric Spaces & Compact Quantum Groups

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1 Compact Quantum Metric Spaces

(X, d) – compact metric space,

$f \in \mathrm{C}(X)$,

$$L(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} \in [0, +\infty].$$

Then

- $L(f)$ is the Lipschitz constant of f,
- L is a seminorm on $\mathrm{C}(X)$,
- The data $(\mathrm{C}(X), L)$ suffices to recover (X, d):

$$d(x, y) = \sup_{L(f) \leq 1} |f(x) - f(y)|,$$

(X, d) – as before, $f \in \mathrm{C}(X)$.

$$L(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} \in [0, +\infty]$$

- enough to take restriction of L to $\mathrm{C}^{\mathbb{R}}(X)$ or to the space A of $\mathbb{R}$-valued Lipschitz functions on X,

- the formula

$$\rho_L(\mu, \nu) = \sup_{L(f) \leq 1} \left| \int_X f \, d\mu - \int_X f \, d\nu \right|,$$

  gives a metric on $\mathrm{Prob}(X) =$ state space of A,

- ρ_L gives $\mathrm{Prob}(X)$ its weak* topology.

Comment: A is an **order unit space**, i.e. it is a partially ordered real vector space with an order unit I such that

$(i)\, \forall\, a \in A \quad \exists\, r \in \mathbb{R} \quad a \leq rI$,

$(ii)\, (\forall\, r > 0 \quad a \leq rI) \Rightarrow (a \leq 0)$.

An order unit space has canonical norm $\| \cdot \|$ and has a state space:

$$S(A) = \{\omega \in A^*_+ :\ \|\omega\| = 1\}.$$

<u>**Definition:**</u> A **compact quantum metric space** is a pair (A, L) such that

1. A is an order unit space (with unit I),
2. L is a seminorm on A with values in $[0, +\infty]$,
3. $(L(a) = 0) \Leftrightarrow (a \in \mathbb{R}I)$,
4. the metric

$$\rho_L(\mu, \nu) = \sup_{L(a) \leq 1} |\mu(a) - \nu(a)|$$

gives $S(A)$ its weak* topology.

A seminorm L on an order unit space A is called a **Lip-norm** iff (A, L) is a compact quantum metric space.

2 Examples

2.1 From group actions

B – unital C*-algebra, G – compact group,

α – ergodic action of G on B,

ℓ – continuous length function on G,

$$L(a) = \sup_{x \neq e} \frac{\|\alpha_x(a) - a\|}{\ell(x)}.$$

Then L is a Lip-norm on $A = B_{\mathrm{s.a.}}$.

- noncommutative tori,
- matrix algebras.

2.2 Dirac operators

A – order unit space, $A \subset \mathrm{B}(H)_{\text{s.a.}}$.

$D^* = D$ – operator on H,

$$L(a) = \|[a, D]\|.$$

- Is L a Lip-norm?
- Every Lip-norm comes from such a construction.

$A = C_r^*(\Gamma) \subset \mathrm{B}(L^2(\Gamma))$,

ℓ – length function on Γ,

$D =$ multiplication by ℓ.

If $\Gamma = \mathbb{Z}^d$ or Γ – hyperbolic then $(A_{\text{s.a.}}, L)$ is a CQMS.

3 Quantum group invariance

3.1 Small metric spaces

Theodor Banica: Finite space X with a metric d and an action

$$v\colon \mathrm{Fun}(X) \longrightarrow H \otimes \mathrm{Fun}(X)$$

of a quantum group (H, Δ) which preserves the metric in the sense that if $X = \{x_1, \ldots, x_N\}$ and

$$v(\delta_{x_k}) = \sum_{l=1}^{N} v_{k,l} \otimes \delta_{x_l},$$

$$d_{k,l} = d(x_k, x_j) I_H$$

then (v_{ij}) commutes with (d_{ij}).

3.2 Invariant Dirac operators

Chakraborty & Pal: $(A, \Delta) = S_qU(2)$,

H – the GNS space of the Haar measure

Problem: find all $D^* = D$ on H such that (A, H, D) is a spectral triple and D commutes with $\widehat{A}$.

SOLVED!

Problem: Is

$$L(a) = \|[a, D]\|$$

a Lip-norm?

3.3 Invariant metrics on compact groups

G – compact group,

d – metric which gives G its original topology,

We have (for $f \in \mathrm{C}(G)$)

$$L(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

One can also compute

$$L'(f) = \sup_{x \neq y} \frac{\|f(\cdot x) - f(\cdot y)\|}{d(x, y)}$$

Then $L \leq L'$ and

$$\left(\begin{array}{c} d \text{ is left} \\ \text{invariant} \end{array} \right) \Longleftrightarrow (\ L' = L\ )$$

3.4 Invariant metrics on compact quantum groups

<u>**Definition:**</u> (A, Δ) – compact quantum group,

L – Lip-norm on $A_{\text{s.a.}}$

L has to be lower semicontinuous,

L is left invariant if

$$L(a) = \sup_{\mu \neq \nu} \frac{\|\mu * a - \nu * a\|}{\rho_L(\mu, \nu)}.$$

(A, Δ) – compact quantum group,

L – lower semicontinuous Lip-norm on $A_{\text{s.a.}}$,

$$L'(a) = \sup_{\mu \neq \nu} \frac{\|a * \mu - a * \nu\|}{\rho_L(\mu, \nu)}.$$

Then L' is a Lip-norm on $A_{\text{s.a.}}$.

Conjecture: The sequence of CQMSs:

$$(A_{\text{s.a.}}, L), (A_{\text{s.a.}}, L'), (A_{\text{s.a.}}, L''), \ldots$$

converges to a CQMS

$$(A_{\text{s.a.}}, L^{(\infty)})$$

with $L^{(\infty)}$ left invariant.

Chakraborty: There is a Lip-norm on $A_{\text{s.a.}}$ for A – the C* algebra of functions on $S_qU(2)$.

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