Examples of compact quantum groups $\mathbb{G}$ with $L^\infty(\mathbb{G})$ a factor

Noncommutative harmonic analysis and quantum groups

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- We will only consider von Neumann algebras acting on separable Hilbert spaces (countably decomposable/with separable predual).
- A von Neumann algebra M is a **factor** if its center is trivial.
- A factor M is of type I if M contains a minimal non-zero projection.
- A factor M is of type II if M contains a non-zero finite projection (not equivalent to its proper subprojection), but does not contain a non-zero minimal projection.
- A factor M is of type III if M does not contain a non-zero finite projection.

Fact

A factor of type I *is isomorphic to* $\mathrm{B}(H)$ *for some* H*. Thus we can classify factors of type* I *into types* I_n ($n \in \mathbb{N} \cup \{\infty\}$) *with* $n = \dim H$*.*

Fact

A factor is of type III *iff it does not admit a non-zero semifinite tracial weight.*

Terminology

Let M be a factor and assume that M is not of type III.

- If M does not admit a tracial state and is of type I then it is of type I_∞.
- If M does not admit a tracial state and is of type II then it is of type II_∞.
- If M admits a tracial state and is infinite dimensional then it is of type II_1.
- A finite-dimensional factor is of type I_n and it admits a tracial state.

- A von Neumann algebra M is **hyperfinite** if it contains a sequence of finite-dimensional subalgebras $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$ such that $\bigcup\limits_{n=1}^{\infty} \mathcal{A}_n$ is dense in M.
- The property of being hyperfinite is equivalent (by a difficult theorem of Connes) to the property of being **injective**, i.e. being an injective object in an appropriate category. This is also equivalent to w*-CPAP.

Theorem

Up to isomorphism there exists only one injective factor of type I_∞, II_1, *and* II_∞.

- The situation with type III factors is more complicated.

Theorem

For $n \in \mathbb{N}$ let M_n be a factor equipped with a faithful tracial weight $\boldsymbol{\tau}_n$ represented on the G.N.S. Hilbert space H_{ω_n} of a faithful non-tracial state ω_n. Then

- *for each n we have $\omega_n = \boldsymbol{\tau}(\cdot\, h_n)$ for some strictly positive operator h_n affiliated with M_n.*

Moreover, letting $\mathsf{M} = \bigotimes\limits_{n=1}^{\infty}(\mathsf{M}_n, \omega_n)$ be the closure of $\mathrm{alg}\text{-}\bigotimes\limits_{n=1}^{\infty}\mathsf{M}_n$ acting on the Hilbert space $\bigotimes\limits_{n=1}^{\infty}(H_{\omega_n}, \Omega_n)$, we additionally have:

- M *is a factor,*
- M *is of type* III *if and only if* $\exists\, t \in \mathbb{R}\ \sum\limits_{n=1}^{\infty}\bigl(1 - |\omega_n(h_n^{\mathrm{i}t})|\bigr) = +\infty$,
- *if each M_n is injective then so is M.*

Connes invariant $T(\mathsf{M})$

Let M be a factor. We define $T(\mathsf{M}) = \{t \in \mathbb{R} \mid \sigma^{\varphi}_t \in \operatorname{Inn}(\mathsf{M})\}$. Then

- $T(\mathsf{M})$ is an invariant of M,
- M is of type III iff $T(\mathsf{M}) \neq \mathbb{R}$.

Connes invariant $S(\mathsf{M})$

Let M be a factor. We define $S(\mathsf{M}) = \bigcap\limits_{\varphi} \operatorname{Sp} \nabla_{\varphi}$. Then

- $S(\mathsf{M})$ is an invariant of M,
- if M is of type III then $S(\mathsf{M})$ must be one of the following:
  - $S(\mathsf{M}) = \{0\} \cup \{\lambda^n \mid n \in \mathbb{Z}\}$ for some $\lambda \in]0, 1[$, type III_λ
  - $S(\mathsf{M}) = \{0, 1\}$, type III_0
  - $S(\mathsf{M}) = \mathbb{R}_{\geqslant 0}$. type III_1
- For $\lambda > 0$ there is only one injective factor of type III_λ.
- For $s \in S(\mathsf{M})$ and $t \in T(\mathsf{M})$ we have $s^{\mathrm{i}t} = 1$.

- $\mathbb{G}$ – a compact quantum group,
- $\mathrm{C}(\mathbb{G})$ – the associated (unital) C*-algebra,
- $\boldsymbol{h}$ – the Haar measure,
- $\mathsf{L}^2(\mathbb{G})$ – the G.N.S. Hilbert space for $\boldsymbol{h}$,
- $\pi_{\boldsymbol{h}}$ – the G.N.S. representation of $\mathrm{C}(\mathbb{G})$ on $\mathsf{L}^2(\mathbb{G})$,
- $\mathsf{L}^\infty(\mathbb{G}) = \pi_{\boldsymbol{h}}\bigl(\mathrm{C}(\mathbb{G})\bigr)''$.

Example

If $\mathbb{G} = \widehat{\Gamma}$ for a discrete i.c.c. group Γ then $\mathsf{L}^\infty(\mathbb{G})$ is a factor of type II_1. We know this because we can directly see that the center of $\mathsf{L}^\infty(\mathbb{G}) = \mathrm{vN}(\Gamma)$ is trivial and there is the tracial state

$$\mathrm{vN}(\Gamma) \ni x \longmapsto \langle \delta_e | x\delta_e \rangle \in \mathbb{C}$$

(δ_e — vector of the standard basis of $\ell^2(\Gamma) = \mathsf{L}^2(\mathbb{G})$).

Theorem (Jacek Krajczok & P.M.S.)

There does not exist a compact quantum group $\mathbb{G}$ *such that* $\mathsf{L}^\infty(\mathbb{G})$ *is a factor of type* I *of dimension strictly grater than one.*

- There are two important (commuting) one-parameter groups of automorphisms of $\mathsf{L}^\infty(\mathbb{G})$:
  - the **modular group** of ***h*** denoted by $(\sigma^{\boldsymbol{h}}_t)_{t\in\mathbb{R}}$,
  - the **scaling group** of $\mathbb{G}$ denoted by $(\tau^{\mathbb{G}}_t)_{t\in\mathbb{R}}$.
- In terms of the Woronowicz functionals $\{f_z\}_{z\in\mathbb{C}}$ they are

$$\begin{aligned}\sigma^{\boldsymbol{h}}_t(a) &= f_{\mathrm{i}t} * a * f_{\mathrm{i}t}\\ \tau^{\mathbb{G}}_t(a) &= f_{\mathrm{i}t} * a * f_{-\mathrm{i}t}\end{aligned}, \qquad t\in\mathbb{R},\ a\in\mathrm{Pol}(\mathbb{G}).$$

- Once we fix for any $\alpha\in\mathrm{Irr}\,\mathbb{G}$ a unitary representation $U^\alpha\in\alpha$ and an orthonormal basis of H_α diagonalizing the corresponding ρ-operator:

$$\rho_\alpha = \mathrm{diag}\big(\rho_{\alpha,1},\dots,\rho_{\alpha,\dim\alpha}\big),$$

the modular and scaling groups are given by

$$\begin{aligned}\sigma^{\boldsymbol{h}}_t(U^\alpha_{i,j}) &= \rho^{\mathrm{i}t}_{\alpha,i}U^\alpha_{i,j}\rho^{\mathrm{i}t}_{\alpha,j}\\ \tau^{\mathbb{G}}_t(U^\alpha_{i,j}) &= \rho^{\mathrm{i}t}_{\alpha,i}U^\alpha_{i,j}\rho^{-\mathrm{i}t}_{\alpha,j}\end{aligned}, \qquad t\in\mathbb{R},\ i,j\in\{1,\dots,\dim\alpha\}.$$

- Let $\{\mathbb{H}_n\}_{n\in\mathbb{N}}$ be a sequence of compact quantum groups.
- Let Ω_n be the G.N.S. cyclic vector for the Haar measure $\boldsymbol{h}_n$ of $\mathbb{H}_n$.
- Then the von Neumann algebra

$$\mathsf{M}=\bigotimes_{n=1}^{\infty}\bigl(\mathsf{L}^{\infty}(\mathbb{H}_n),\boldsymbol{h}_n\bigr)$$

  carries a comultiplication defining a compact quantum group such that the inclusions $\mathsf{L}^{\infty}(\mathbb{H}_n)\hookrightarrow\mathsf{M}$ intertwine comultiplications.
- We denote this compact quantum group by $\bigtimes_{n=1}^{\infty}\mathbb{H}_n$.
- The Haar measure of $\bigtimes_{n=1}^{\infty}\mathbb{H}_n$ is $\bigotimes_{n=1}^{\infty}\boldsymbol{h}_n$.

- Let $\mathbb{H}=\bigtimes\limits_{n=1}^{\infty}\mathbb{H}_n$.
- For any $\alpha_1\in\operatorname{Irr}\mathbb{H}_{n_1}$ and $\alpha_2\in\operatorname{Irr}\mathbb{H}_{n_2}$ and $U^{\alpha_i}\in\alpha_i$ we define $\alpha_1\boxtimes\alpha_2$ as the class of

$$\begin{aligned}\big(U^{\alpha_1}\big)_{13}\big(U^{\alpha_2}\big)_{24}&\in\mathrm{B}(H_{\alpha_1})\otimes\mathrm{B}(H_{\alpha_2})\otimes\mathsf{L}^{\infty}(\mathbb{H}_{n_1})\otimes\mathsf{L}^{\infty}(\mathbb{H}_{n_2})\\&\subset\mathrm{B}(H_{\alpha_1}\otimes H_{\alpha_2})\otimes\mathsf{L}^{\infty}(\mathbb{H}).\end{aligned}$$

- Similarly we define $\alpha_1\boxtimes\cdots\boxtimes\alpha_N$ for $\alpha_i\in\operatorname{Irr}\mathbb{H}_{n_i}$ with pairwise different $n_1,\dots,n_N$.
- Such "exterior tensor products" exhaust all classes of irreps of $\mathbb{H}$.

- Fix a compact quantum group $\mathbb{H}$.
- Let Γ be a discrete group acting on $\mathsf{L}^\infty(\mathbb{H})$ by automorphisms of $\mathbb{H}$.
- The von Neumann algebra $\Gamma \ltimes \mathsf{L}^\infty(\mathbb{H})$ carries a unique comultiplication such that

$$\mathsf{L}^\infty(\mathbb{H}) \hookrightarrow \Gamma \ltimes \mathsf{L}^\infty(\mathbb{H}) \quad \text{and} \quad \mathrm{vN}(\Gamma) \hookrightarrow \Gamma \ltimes \mathsf{L}^\infty(\mathbb{H})$$

  intertwine comultiplications.
- With this comultiplication $\Gamma \ltimes \mathsf{L}^\infty(\mathbb{H})$ describes a compact quantum group which we denote $\Gamma \bowtie \mathbb{H}$.
- Irreps of $\Gamma \bowtie \mathbb{H}$ are all of the form

$$(\mathbb{1} \otimes u_\gamma)\big((\mathrm{id} \otimes \alpha)U^\lambda\big),$$

  where
  - $\alpha\colon \mathsf{L}^\infty(\mathbb{H}) \to \ell^\infty(\Gamma) \otimes \mathsf{L}^\infty(\mathbb{H}) \subset \Gamma \ltimes \mathsf{L}^\infty(\mathbb{H})$ is the action,
  - $\{u_\gamma\}_{\gamma\in\Gamma}$ are the unitaries implementing the action,
  - $\lambda \in \mathrm{Irr}\,\mathbb{H}$ and $U^\lambda \in \mathrm{B}(H_\lambda) \otimes \mathsf{L}^\infty(\mathbb{H})$ is a representative of λ.

Example

Let $\mathbb{H}=\mathrm{SU}_q(2)$ and $\Gamma=\mathbb{Q}$. Fix $\nu\in\mathbb{R}\setminus\{0\}$, $q\in\,]-1,1[\,\setminus\{0\}$ and let $\gamma\in\Gamma$ act on $\mathsf{L}^\infty(\mathbb{H})$ by $\tau^{\mathbb{H}}_{\nu\gamma}$. We will denote the resulting compact quantum group $\mathbb{Q}\bowtie\mathrm{SU}_q(2)$ by $\mathbb{H}_{\nu,q}$.

Theorem (Krajczok-Wasilewski)

We have

- $\mathbb{H}_{\nu,q}$ *is a co-amenable compact quantum group.*
- *If* $\nu\log|q|\notin\pi\mathbb{Q}$ *then* $\mathsf{L}^\infty(\mathbb{H}_{\nu,q})$ *is a factor.*
- *Since there is a tracial weight on* $\mathsf{L}^\infty\bigl(\mathrm{SU}_q(2)\bigr)$ *invariant under the scaling group, the algebra* $\mathsf{L}^\infty(\mathbb{H}_{\nu,q})$ *is not of type* III.

Furthermore, assuming $\nu\log|q|\notin\pi\mathbb{Q}$*, we have*

- $\tau^{\mathbb{H}_{\nu,q}}_t$ *is trivial iff* $t\in\frac{\pi}{\log|q|}\mathbb{Z}$*, so* $\mathbb{H}_{\nu,q}$ *is not of Kac type,*
- *consequently* $\mathsf{L}^\infty(\mathbb{H}_{\nu,q})$ *is a factor of type* II_∞.

Theorem

For any $\lambda \in]0,1[$ there exists a compact quantum group $\mathbb{G}$ such that $\mathsf{L}^\infty(\mathbb{G})$ is the injective factor of type III_λ.

- We take $q = \sqrt{\lambda}$ and ν such that $\nu \log q \notin \pi\mathbb{Q}$.
- Put $\mathbb{G} = \mathop{\times}\limits_{n=1}^{\infty} \mathbb{H}_{\nu,q}$.
- Then $\mathsf{L}^\infty(\mathbb{G})$ is an injective factor.
- $T\bigl(\mathsf{L}^\infty(\mathbb{G})\bigr) = \frac{\pi}{\log q}\mathbb{Z}$, so $\mathsf{L}^\infty(\mathbb{G})$ is of type III.
- $S\bigl(\mathsf{L}^\infty(\mathbb{G})\bigr) = \overline{\bigl\{\lambda_1 \cdots \lambda_N \,\big|\, N \in \mathbb{N},\ \lambda_i \in \mathrm{Sp}(\nabla_{\boldsymbol{h}_{\nu,q}}),\ i \in \{1,\dots,N\}\bigr\}}$.
- $\mathrm{Sp}(\nabla_{\boldsymbol{h}_{\nu,q}}) = \{0\} \cup q^{2\mathbb{Z}}$, so $\mathsf{L}^\infty(\mathbb{G})$ is of type III_{q^2}.

Theorem

There exists a family $\{\mathbb{G}_s\}_{s\in]0,1[}$ of compact quantum groups such that $\{\mathsf{L}^\infty(\mathbb{G}_s)\}_{s\in]0,1[}$ are pairwise non-isomorphic injective factors of type III_0.

- Let $(q_n)_{n\in\mathbb{N}}$ be the sequence

$$\bigl(\underbrace{\exp(-\pi 1!),\dots,\exp(-\pi 1!)}_{l_1\text{ times}},\underbrace{\exp(-\pi 2!),\dots,\exp(-\pi 2!)}_{l_2\text{ times}},\dots\bigr),$$

  where $l_k=\bigl\lfloor\exp(2\pi k!)k^{2s-1}\bigr\rfloor$.
- For each n choose $\nu_n\in\mathbb{R}\setminus\{0\}$ such that $\nu_n\log q_n\notin\pi\mathbb{Q}$.
- $\mathbb{G}_s=\bigtimes_{n=1}^{\infty}\mathbb{H}_{\nu_n,q_n}$.
- $\mathsf{L}^\infty(\mathbb{G}_s)$ is an injective factor.

THEOREM

There exists a family $\{\mathbb{G}_s\}_{s\in]0,1[}$ of compact quantum groups such that $\{\mathsf{L}^\infty(\mathbb{G}_s)\}_{s\in]0,1[}$ are pairwise non-isomorphic injective factors of type III_0.

- For each s we have $\mathbb{Q}\subset T\bigl(\mathsf{L}^\infty(\mathbb{G}_s)\bigr)$.
- Put
$$t_s=\sum_{p=1}^{\infty}\frac{\lfloor p^{1-s}\rfloor}{p!}.$$
- The map $s\mapsto t_s$ is strictly decreasing.
- Then $t_{s'}\in T\bigl(\mathsf{L}^\infty(\mathbb{G}_s)\bigr)$ iff $s'>s$.
- This shows that
    - $\mathsf{L}^\infty(\mathbb{G}_s)$ is not isomorphic to $\mathsf{L}^\infty(\mathbb{G}_{s'})$ if $s\neq s'$,
    - $\mathsf{L}^\infty(\mathbb{G})$ is of type III_0.

Theorem

There exists a compact quantum group $\mathbb{G}$ such that $\mathsf{L}^\infty(\mathbb{G})$ is the injective factor of type III_1.

- Let $q_1, q_2 \in \left]-1, 1\right[\setminus \{0\}$ be such that $\frac{\pi}{\log|q_1|}\mathbb{Q} \cap \frac{\pi}{\log|q_2|}\mathbb{Q} = \{0\}$.
- Choose ν_i, so that $\nu_i \log|q_i| \notin \pi\mathbb{Q}$ ($i = 1, 2$).
- Put
$$\mathbb{H}_n = \begin{cases} \mathbb{H}_{\nu_1, q_1} & n \text{ is odd} \\ \mathbb{H}_{\nu_2, q_2} & n \text{ is even} \end{cases}.$$
- $\mathbb{G} = \bigtimes\limits_{n=1}^{\infty} \mathbb{H}_n$.
- We have $T\bigl(\mathsf{L}^\infty(\mathbb{G})\bigr) = \{0\}$, $S\bigl(\mathsf{L}^\infty(\mathbb{G})\bigr) = \mathbb{R}_{\geqslant 0}$.

DEFINITION

Let $\mathbb{G}$ be a (locally) compact quantum group. Define

$$T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \left\{ t \in \mathbb{R} \,\middle|\, \tau^{\mathbb{G}}_t \in \mathrm{Inn}\big(\mathsf{L}^{\infty}(\mathbb{G})\big) \right\}.$$

- $T^{\tau}_{\mathrm{Inn}}(\mathbb{G})$ is an invariant of $\mathbb{G}$.
- $T^{\tau}_{\mathrm{Inn}}(\mathbb{G})$ is a subgroup of $\mathbb{R}$.
- $T^{\tau}_{\mathrm{Inn}}(\mathbb{G} \times \mathbb{H}) = T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) \cap T^{\tau}_{\mathrm{Inn}}(\mathbb{H})$.

EXAMPLES

- $T^{\tau}_{\mathrm{Inn}}\big(\mathrm{SU}_q(2)\big) = \frac{\pi}{\log|q|}\mathbb{Z}$,
- $T^{\tau}_{\mathrm{Inn}}(\mathbb{H}_{\nu,q}) = \nu\mathbb{Q} + \frac{\pi}{\log|q|}\mathbb{Z}$.

THEOREM

For each $\lambda \in \,]0,1[$ there exists an uncountable family $\{\mathbb{K}^\lambda_j\}_{j\in\mathbb{J}}$ of pairwise non-isomorphic compact quantum groups such that $\mathsf{L}^\infty(\mathbb{K}^\lambda_j)$ is the injective factor of type III_λ.

- Take $q = \sqrt{\lambda}$, ν such that $\nu \log q \notin \pi\mathbb{Q}$, and let $\Gamma_j = \alpha_j \frac{\pi}{\log q}\mathbb{Z}$, where $\{1\} \cup \{\alpha_j\}_{j\in\mathbb{J}}$ is a basis of $\mathbb{R}$ over $\mathbb{Q}$.
- Let $\mathbb{G} = \mathop{\times}\limits_{n=1}^{\infty} \mathbb{H}_{\nu,q}$ and let Γ_j act on $\mathbb{G}$ by the scaling automorphisms.
- Put $\mathbb{K}^\lambda_j = \Gamma_j \bowtie \mathbb{G}$.
- $\mathsf{L}^\infty(\mathbb{K}^\lambda_j)$ is the injective factor of type III_λ.
- $T^{\tau}_{\mathrm{Inn}}(\mathbb{K}^\lambda_j) = \Gamma_j + \frac{\pi}{\log q}\mathbb{Z}$.
- For $j \neq j'$ we have $\Gamma_j + \frac{\pi}{\log q}\mathbb{Z} \neq \Gamma_{j'} + \frac{\pi}{\log q}\mathbb{Z}$.

Theorem

There exists uncountably many pairwise non-isomorphic compact quantum groups $\mathbb{K}$ with $\mathsf{L}^{\infty}(\mathbb{K})$ the injective factor of type III_1.

- Start with the example of $\mathbb{G}$ with $\mathsf{L}^{\infty}(\mathbb{G})$ the injective factor of type III_1 we discussed earlier:
  - Let $q_1, q_2 \in]-1, 1[\setminus \{0\}$ be such that $\frac{\pi}{\log|q_1|}\mathbb{Q} \cap \frac{\pi}{\log|q_2|}\mathbb{Q} = \{0\}$.
  - Choose ν_i, so that $\nu_i \log|q_i| \notin \pi\mathbb{Q}$ ($i = 1, 2$).
  - Put
$$\mathbb{H}_n = \begin{cases} \mathbb{H}_{\nu_1,q_1} & n \text{ is odd} \\ \mathbb{H}_{\nu_2,q_2} & n \text{ is even} \end{cases}.$$
  - $\mathbb{G} = \mathop{\times}\limits_{n=1}^{\infty} \mathbb{H}_n$.
- Let Γ be a countable subgroup of $\mathbb{R}$ and let Γ act on $\mathbb{G}$ by the scaling automorphisms.
- Put $\mathbb{K} = \Gamma \ltimes \mathbb{G}$.
- Then $\mathsf{L}^{\infty}(\mathbb{K})$ is an injective factor and $S\big(\mathsf{L}^{\infty}(\mathbb{K})\big) = \mathbb{R}_{\geqslant 0}$.
- Furthermore $T^{\tau}_{\mathrm{Inn}}(\mathbb{K}) = \Gamma$.

Thank you for your attention.