Quantum groups

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Outline

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    - Actions & homogeneous spaces
    - Baum-Connes conjecture

Let $\mathbb{G}$ be a locally compact group

- $\mathbb{G}$ (loc. comp. space)
- $m : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{G}$

$$(t, s) \longmapsto ts$$

- $s(tr) = (st)r$
- Right invariant Haar measure

$$\int_{\mathbb{G}} f(ts)\, dt = \int_{\mathbb{G}} f(t)\, dt$$

- $L^2(\mathbb{G})$
- $U \in \mathrm{Rep}(\mathbb{G}, \mathcal{H})$

$$U_t U_s = U_{ts}$$

- $A = \mathrm{C}_0(\mathbb{G})$ (C^*-algebra)
- $\Delta \in \mathrm{Mor}(A, A \otimes A)$

$$\big(\Delta(f)\big)(t, s) = f(ts)$$

- $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$
- $h : A_+ \longrightarrow [0, +\infty]$

$$(h \otimes \mathrm{id})\Delta(f) = h(f)\mathbb{1}$$

- GNS-Hilbert space for h
- $U \in \mathrm{M}\big(\mathcal{K}(\mathcal{H}) \otimes A\big)$

$$(\mathrm{id} \otimes \Delta)(U) = U_{12} U_{13}$$

Assume that $\mathbb{G}$ is abelian

- The Pontryagin dual of $\mathbb{G}$:

$$\widehat{\mathbb{G}} = \left\{\phi : \mathbb{G} \longrightarrow \mathbb{T} \,\middle|\, \phi \text{ is a continuous homomorphism}\right\}$$

  is a locally compact abelian group
- $\mathrm{C}_0\bigl(\widehat{\mathbb{G}}\bigr) \simeq \mathrm{C}^*(\mathbb{G})$ (Fourier/Gelfand transform)
- For $\widehat{A} = \mathrm{C}^*(\mathbb{G})$ we have
  - $\widehat{\Delta} \in \mathrm{Mor}\bigl(\widehat{A}, \widehat{A} \otimes \widehat{A}\bigr)$,
  - $\widehat{h} : \widehat{A}_+ \longrightarrow [0, \infty]$
  - etc.
- $\widehat{A} = \mathrm{C}^*(\mathbb{G})$ with all its additional structure exists for non-abelian $\mathbb{G}$.
- How to generalize $\widehat{\widehat{\mathbb{G}}} \simeq \mathbb{G}$? $\bigl(\widehat{A}, \widehat{\Delta}\bigr)$ is a quantum group

S.L. Woronowicz 1987

- $q \in [-1, 1] \setminus \{0\}$
- $A :=$ C^*-algebra generated by α and γ such that

$$\begin{aligned} \alpha\gamma = q\gamma\alpha, &\qquad \alpha^*\alpha + \gamma^*\gamma = \mathbb{1}, \\ \gamma^*\gamma = \gamma\gamma^*, &\quad \alpha\alpha^* + q^2\gamma^*\gamma = \mathbb{1} \end{aligned}$$

- $\Delta : A \longrightarrow A \otimes A$

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \qquad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$$

- For $q = 1$, $A = \mathrm{C}\big(\mathrm{SU}(2)\big)$ with Δ as before
- For $q \neq 1$, we obtain the quantum group $\mathrm{S}_q\mathrm{U}(2)$
- $\mathrm{S}_q\mathrm{U}(2) = (A, \Delta)$ is not commutative nor co-commutative

Some things we won't talk about

- Hopf algebras
  - $+$ Source of simple examples
  - $+$ Fine with compact quantum groups
  - $-$ Problems with duality
  - $-$ Not applicable to sophisticated examples (non-compact)
- Multiplier Hopf algebras
  - $+$ Nice framework for duality
  - $+$ Good "laboratory"
  - $-$ Not applicable for "topologically nontrivial" situations
- von Neumann algebraic quantum groups
  - $+$ Very powerfull approach
  - $\pm$ Technically complicated
  - $+$ Equivalent to C^*-algebraic approach

Compact quantum groups

- $\mathbb{G} = (A, \Delta)$ is a compact quantum group if
  - A is a unital C^*-algebra
  - $\Delta \in \mathrm{Mor}(A, A \otimes A), \quad (\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$
  - $(A \otimes \mathbb{1})\Delta(A), \Delta(A)(\mathbb{1} \otimes A) \subset_{\text{DENSE}} A \otimes A$
- For compact quantum groups we have
  - Examples
  - Proof of existence of Haar measure (Woronowicz)
  - Representation theory (Peter-Weyl-Woronowicz thm)
  - Actions & homogeneous spaces (Podleś)
  - Differential calculi (Woronowicz, Podleś)
  - Duality ($\widehat{\mathbb{G}}$ is a discrete quantum group)

X — loc. comp. Then

$$\Big(X \text{ is compact}\Big) \Updownarrow \Big(\mathrm{C}_0(X) \text{ is unital}\Big)$$

Non-compact quantum groups

- $\mathbb{G} = (A, \Delta)$ is a locally compact quantum group if
  - A is a C^*-algebra with two faithful K.M.S. weights φ and ψ
  - $\Delta \in \mathrm{Mor}(A, A \otimes A)$, $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$
  - $(A \otimes \mathbb{1})\Delta(A), \Delta(A)(\mathbb{1} \otimes A) \subset_{\mathrm{DENSE}} A \otimes A$
  - $(\mathrm{id} \otimes \varphi)\Delta(a) = \varphi(a)\mathbb{1}, \quad (\psi \otimes \mathrm{id})\Delta(a) = \psi(a)\mathbb{1},$
- We have exciting examples:
  - quantum $E(2)$ (Woronowicz)
  - various quantum Lorentz ($\mathrm{SL}(2, \mathbb{C})$) groups (Podleś-Woronowicz, Woronowicz-Zakrzewski, Kasprzak)
  - quantum "$ax + b$" and "$az + b$" groups (Woronowicz, PMS)
  - quantum $\mathrm{GL}(2, \mathbb{C})$ (Pusz)
  - bicrossed products – examples from number theory (Baas-Skandalis-Vaes)

Haar measure challenge

- Let $\mathbb{G} = (A, \Delta)$ be a locally compact quantum group
- Existence of Haar measure(s) is part of the definition
- Alternative definition and proof of existence of Haar measures only in special cases: compact, discrete
- Possible to construct $\mathbb{G}$ from multiplicative unitaries (not using Haar measures)
- Possible definition: $\mathbb{G}$ is a quantum group when it comes from an appropriate multiplicative unitary
- In all examples Haar measures are there
- We have formula for invariant measure in general case (no guarantee it works)
- There exists the class of Haar measure in appropriate sense (PMS-Woronowicz)

Actions & homogeneous spaces challenge

- Actions and Homogeneous spaces for compact quantum groups are well studied (Podleś)
- For locally compact quantum groups there is the powerfull approach of S. Vaes
    - Complicated technically
    - Some definitions – not intuitive
    - Few well studied examples (beyond classical ones)

Baum-Connes conjecture challenge

- Let Γ be a group
- We have the assembly map

$$\mu : KK_j^{\Gamma}\big(C_0(\underline{E\Gamma}), \mathbb{C}\big) \longrightarrow K_j\big(\mathrm{C}_r^*(\Gamma)\big)$$

- On the right hand side we have the topological K-theory of the quantum space whose algebra of C_0-functions is $\mathrm{C}_r^*(\Gamma)$
- This quantum space happens to be a quantum group, whose dual is classical (abelian quantum group)
- This suggests we should formulate the conjecture for quantum groups
- Attempt was made using Meyer-Nest approach (not absolutely straightforward generalization)
- The conjecture was proved for the dual of $\mathrm{S}_q\mathrm{U}(2)$ (Voigt)