# Subgroups of Quantum Groups, the Center and Inner Automorphisms

Non-commutative Workshop

**Kraków**

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# Plan of talk

1. Locally compact quantum groups
2. Quantum subgroups
3. The center
4. Quantum group of inner automorphisms
5. The exact sequence

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  - $\varphi$ and $\psi$ are given by integration w.r.t. left and right Haar measures.

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# The antipode

- A l.c.q.g. $\mathbb{G}$ has an **antipode** $S$ which is a densely defined map $L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ roughly characterized by the fact that for all $\omega \in \mathrm{B}(L^2(\mathbb{G}))_*$ we have $(\omega \otimes \mathrm{id})(W) \in \mathrm{D}(S)$ and

$$S\big((\omega \otimes \mathrm{id})(W)\big) = (\omega \otimes \mathrm{id})(W^*).$$

- The antipode has the following unique decomposition:

$$S = R \circ \tau_{\mathrm{i}/2},$$

  where
  - $R$ is a $*$-anti-automorphism of $L^\infty(\mathbb{G})$ (**unitary antipode**),
  - $\tau_{\mathrm{i}/2}$ is the analytic continuation to $t = \mathrm{i}/2$ of a one parameter group $(\tau_t)_{t \in \mathbb{R}}$ of automorphisms of $L^\infty(\mathbb{G})$ such that $\tau_t \circ R = R \circ \tau_t$ for all $t$ (**scaling group**).

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- The image of $\gamma$ as above is $R^{\widehat{\mathbb{G}}}$ and $\tau^{\widehat{\mathbb{G}}}$-invariant.

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- The relative commutant $\mathsf{N}' \cap L^\infty(\mathbb{G})$ plays the role of the algebra $L^\infty(\mathbb{G}/\mathbb{H})$.

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**Definition**

The center $\mathscr{Z}(\mathbb{G})$ of a locally compact quantum group $\mathbb{G}$ is defined as the quantum subgroup corresponding to the largest central Baaj-Vaes subalgebra of $L^\infty(\widehat{\mathbb{G}})$.

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- For a compact semisimple simply connected Lie group $G$ the dual $\widehat{G_q}$ of the **Drinfeld deformation** $G_q$ ($0 < q < 1$) has trivial center.

# INNER AUTOMORPHISMS

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**Theorem**

*Let* $\mathbb{G}$ *be a locally compact quantum group and let* $\mathsf{M} \subset L^\infty(\mathbb{G})$ *be the subalgebra generated by*

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We define the locally compact quantum group $\mathrm{Inn}(\mathbb{G})$ by setting

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- Subgroups $\mathbb{H}$ satisfying the above conditions are called **normal**.

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2. $L^\infty(\mathbb{G}/\mathbb{H})$ *is a Baaj-Vaes subalgebra of* $L^\infty(\mathbb{G})$ *(i.e. $\mathbb{G}/\mathbb{H}$ is a quantum group).*

- Subgroups $\mathbb{H}$ satisfying the above conditions are called **normal**. (This agrees with Wang's definition from 1995.)

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- In particular this gives a more concrete description of $L^\infty(\widehat{\mathscr{Z}(\mathbb{G})})$.
- The conclusion of the theorem above can be rephrased as

$$\{e\} \longrightarrow \mathscr{Z}(\mathbb{G}) \longrightarrow \mathbb{G} \longrightarrow \mathrm{Inn}(\mathbb{G}) \longrightarrow \{e\}.$$