Lattice of idempotent states on a locally compact quantum group

Noncommutative Geometry Seminar,
Instytut Matematyczny Polskiej Akademii Nauk,
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1. IDEMPOTENT STATES
   - Quasi-subgroups
   - Idempotent states, coideals & group-like projections
   - Order on quasi-subgroups
2. LATTICE OPERATIONS
   - Intersection
   - Generation
   - Modular law
3. OPEN QUASI-SUBGROUPS
   - Duality
   - Compact and discrete quantum groups
   - Operations on open quasi-subgroups

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($\boldsymbol{h}_K$ = *the Haar measure on K*, $\mathrm{C}_0(G) = \mathrm{C}_0^{\mathrm{u}}(G)$ *canonically*).

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- Given $\omega \in \mathrm{Idem}(\mathbb{G})$ we will say that ω corresponds to a compact quantum **quasi-subgroup** of $\mathbb{G}$.

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  where $\mathbb{W} \in \mathsf{M}(\mathrm{C}_0(\hat{\mathbb{G}}) \otimes \mathrm{C}_0^{\mathrm{u}}(\mathbb{G})) \subset \mathsf{M}(\mathscr{K}(L^2(\mathbb{G}))) \otimes \mathrm{C}_0^{\mathrm{u}}(\mathbb{G}))$ is the "half-lifted" Kac-Takesaki operator of $\mathbb{G}$.

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- If ω arises as the Haar measure of a compact quantum subgroup $\mathbb{K}$ then
$$\mathsf{N}_\omega = L^\infty(\mathbb{G}/\mathbb{K}).$$

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- $L^2(\mathsf{N}_\omega) \cap L^2(\mathsf{N}_\mu) \neq \{0\}$ *if and only if* $\nu \neq 0$; *moreover, in this case,* ν *is the idempotent state corresponding to the* τ*-invariant integrable left coideal* $\mathsf{N}_\omega \cap \mathsf{N}_\mu$.

- $\mathsf{M} \subset L^\infty(\mathbb{G})$ – coideal. Then $L^2(\mathsf{M}) \neq \{0\}$ iff M is integrable.

- The mapping $\mathrm{Idem}(\mathbb{G}) \times \mathrm{Idem}(\mathbb{G}) \to \mathrm{Idem}_0(\mathbb{G})$

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- We say that $\omega\vee\mu$ corresponds to the quasi-subgroup of $\mathbb{G}$ **generated** by the quasi-subgroups related to ω and μ.

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We have

1. $x \in \mathsf{N}_\omega$ *if and only if* $\Delta(x)(\mathbb{1} \otimes Q_\omega^\perp) = x \otimes Q_\omega^\perp$,
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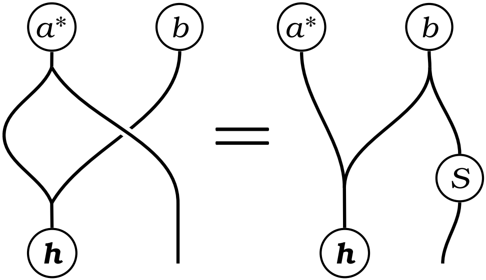

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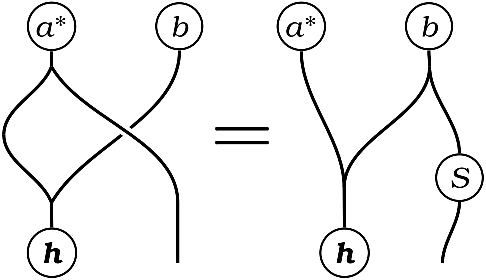

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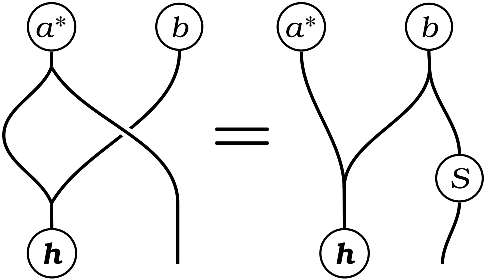

Corollary

Let $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$. Then $Q_\omega^\perp$ is a group-like projection.

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Take arbitrary $x \in \mathsf{N}_\omega$.

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It follows that for any $\mu \in L^{\infty}(\mathbb{G})_{*}$ we have

$$(\mu \otimes \mathrm{id})\big(\Delta(x)\big)\mathcal{Q}_{\omega}^{\perp} = \mu(x)\mathcal{Q}_{\omega}^{\perp} = \mathcal{Q}_{\omega}^{\perp}(\mu \otimes \mathrm{id})\big(\Delta(x)\big).$$

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The result follows from σ-weak linear density of elements of the form $(\mu \otimes \mathrm{id})\big(\Delta(x)\big)$ in N_ω.

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$$\widetilde{\omega}(y) = \frac{\widehat{\boldsymbol{h}}(P_\omega\, y P_\omega)}{\widehat{\boldsymbol{h}}(P_\omega)}, \qquad y \in L^\infty(\widehat{\mathbb{G}}).$$

- Important ingredient of the proof:

$$\widetilde{\mathsf{N}_\omega} = \left\{ y \in L^\infty(\widehat{\mathbb{G}}) \,\middle|\, \widehat{\Delta}(y)(\mathbb{1} \otimes P_\omega) = y \otimes P_\omega \right\}$$

Theorem

Let $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$*. Then there exists a unique* $\widetilde{\omega} \in \mathrm{Idem}_{\mathrm{nor}}(\widehat{\mathbb{G}})$ *such that* $\widetilde{\mathsf{N}_\omega} = \mathsf{N}_{\widetilde{\omega}}$*. Moreover*

- *the mapping*

$$\mathrm{Idem}_{\mathrm{nor}}(\mathbb{G}) \ni \omega \longmapsto \widetilde{\omega} \in \mathrm{Idem}_{\mathrm{nor}}(\widehat{\mathbb{G}})$$

  satisfies $\widetilde{\widetilde{\omega}} = \omega$ *for all* ω*,*
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- The latter is, in fact, an extension of a special case of the former.

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If $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ then N_ω admits a minimal central projection, so it has a finite dimensional direct summand.

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If $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ then N_ω admits a minimal central projection, so it has a finite dimensional direct summand. Also the action of $\mathbb{G}$ on N_ω is ergodic.

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If $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ then N_ω admits a minimal central projection, so it has a finite dimensional direct summand. Also the action of $\mathbb{G}$ on N_ω is ergodic. By theorem on ergodic actions of c.q.g.'s on such von Neumann algebras we have $\dim \mathsf{N}_\omega < +\infty$.

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P_ω is a projection onto the finite-dimensional space $L^2(\mathsf{N}_\omega)$, so $\widehat{\boldsymbol{h}}(P_\omega) < +\infty$.

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THEOREM

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is a normal idempotent state and $\widetilde{\mathsf{N}_\omega} = \mathsf{N}_{\tilde{\omega}}$. It follows that $\omega = \tilde{\tilde{\omega}}$ and so $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$.

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COROLLARY

Let $\mathbb{G}$ *be a compact quantum group and* $\omega \in \mathrm{Idem}(\mathbb{G})$. *Then* N_ω *has a finite dimensional direct summand if and only if* $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$.

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Let $\mathbb{G}$ be a compact quantum group and $\omega \in \mathrm{Idem}(\mathbb{G})$. Then N_ω has a finite dimensional direct summand if and only if $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$.

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If $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ then $\dim \mathsf{N}_\omega < +\infty$

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If $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ then $\dim \mathsf{N}_\omega < +\infty$, so N_ω is a direct sum of matrix algebras.

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Proof

If $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ then $\dim \mathsf{N}_\omega < +\infty$, so N_ω is a direct sum of matrix algebras. Conversely

Corollary

Let $\mathbb{G}$ be a compact quantum group and $\omega \in \mathrm{Idem}(\mathbb{G})$. Then N_ω has a finite dimensional direct summand if and only if $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$.

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If $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ then $\dim \mathsf{N}_\omega < +\infty$, so N_ω is a direct sum of matrix algebras. Conversely, if N_ω has a finite dimensional direct summand

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If $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ then $\dim \mathsf{N}_\omega < +\infty$, so N_ω is a direct sum of matrix algebras. Conversely, if N_ω has a finite dimensional direct summand then by theorem on ergodic actions we have $\dim \mathsf{N}_\omega < +\infty$

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We know that the coideal N_ω is integrable

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We know that the coideal N_ω is integrable, so as $\dim \mathsf{N}_\omega < +\infty$, we have $\boldsymbol{h}(\mathbb{1}) < +\infty$

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Let P be the projection onto $L^{\infty}(\mathsf{N})$. One can demonstrate that $P \in \ell^{\infty}(\widehat{\mathbb{G}})$. Then we show that P is group-like

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Let P be the projection onto $L^\infty(\mathsf{N})$. One can demonstrate that $P \in \ell^\infty(\widehat{\mathbb{G}})$. Then we show that P is group-like: it is enough to see that

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$$\mathsf{N}_{\widetilde{\omega \wedge \mu}} = \widetilde{\mathsf{N}_{\omega \wedge \mu}} = \mathsf{N}_{\widetilde{\omega}} \vee \mathsf{N}_{\mu}$$

Proposition

Let $\omega, \mu \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$*. Then* $\omega \wedge \mu \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ *if and only if* $\tilde{\omega} \vee \tilde{\mu} \neq 0$*. In this case we have*

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