Embeddable quantum homogeneous spaces

Noncommutative Geometry and Quantum Groups
The Fields Institute for Research in Mathematical Sciences

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Faculty of Physics, University of Warsaw

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Plan of talk

1. Locally compact quantum groups
2. Quantum $\mathbb{G}$-spaces
3. Closed quantum subgroups and quotients
4. W*-quantum homogeneous $\mathbb{G}$-spaces
5. Embeddable quantum homogeneous spaces
6. Quotient by the diagonal subgroup

L.C.Q.G.'s

Definition

A **locally compact quantum group** $\mathbb{G}$ consist of a von Neumann algebra M, a normal unital injective map

$$\Delta\colon \mathsf{M} \longrightarrow \mathsf{M} \bar{\otimes} \mathsf{M}$$

such that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, and two n.s.f. weights φ and ψ on M such that

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- Both $L^\infty(\mathbb{G})$ and $L^\infty(\widehat{\mathbb{G}})$ are naturally represented on the GNS Hilbert space of ψ called $L^2(\mathbb{G})$.

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- There is another C^*-algebra $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})$ with a surjection $\Lambda\colon \mathrm{C}_0^{\mathrm{u}}(\mathbb{G}) \to \mathrm{C}_0(\mathbb{G})$ which encodes representation theory of $\widehat{\mathbb{G}}$; $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})$ is called the **universal version** of $\mathrm{C}_0(\mathbb{G})$.

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- By analogy with group C^*-algebras, $C_0(\mathbb{G})$ is often called the **reduced** C^*-algebra describing $\mathbb{G}$.

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  are relevant (and desirable).

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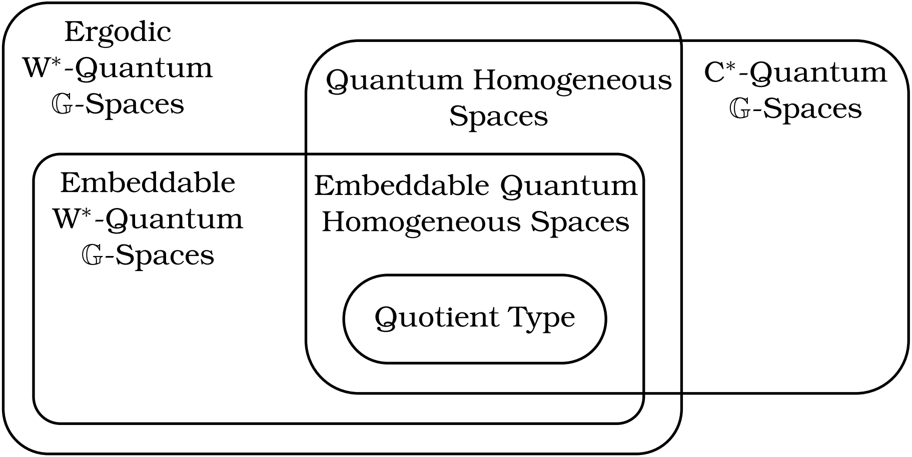

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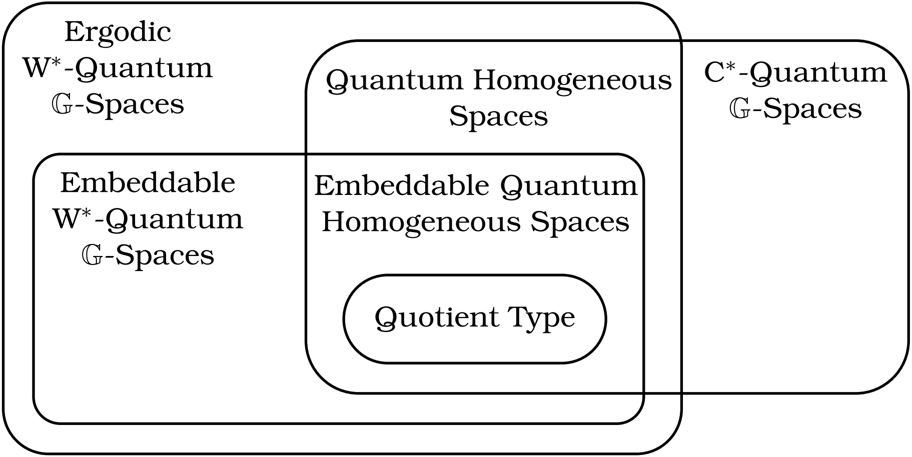

- Many classes of objects, some relations unclear

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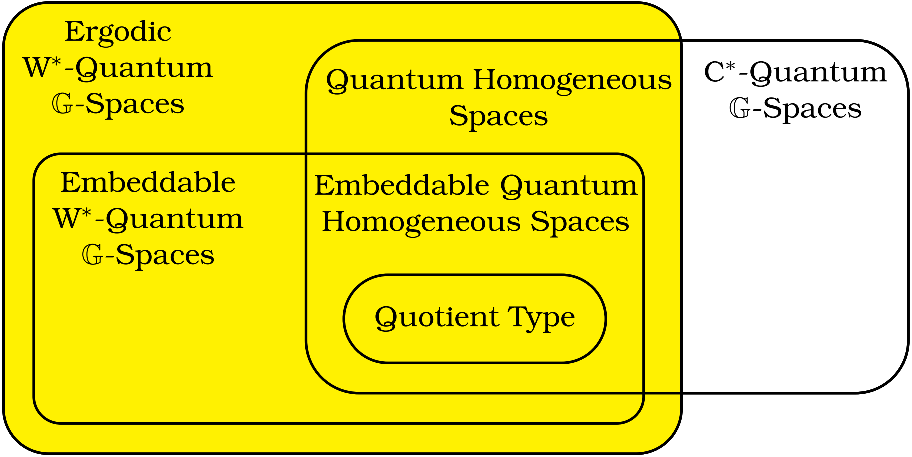

- Von Neumann algebra language, $\alpha(x) = \mathbb{1} \otimes x \;\Rightarrow\; x \in \mathbb{C}\mathbb{1}$

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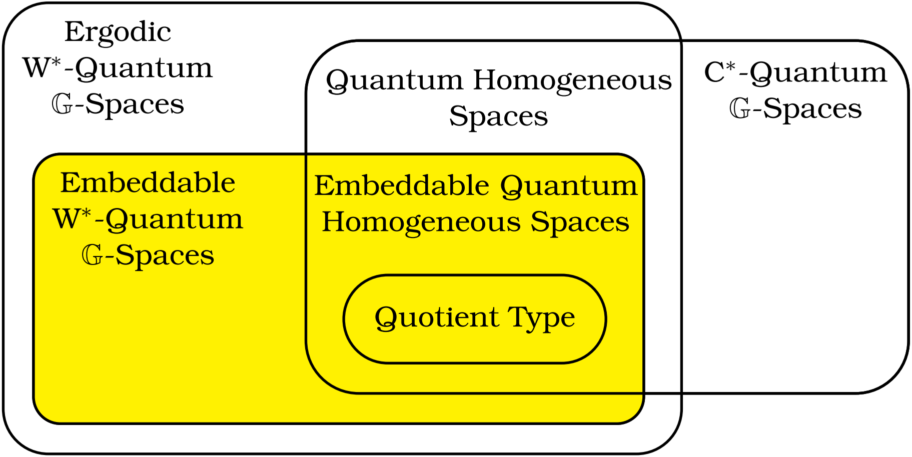

- Left coideals in $L^\infty(\mathbb{G})$, co-duality

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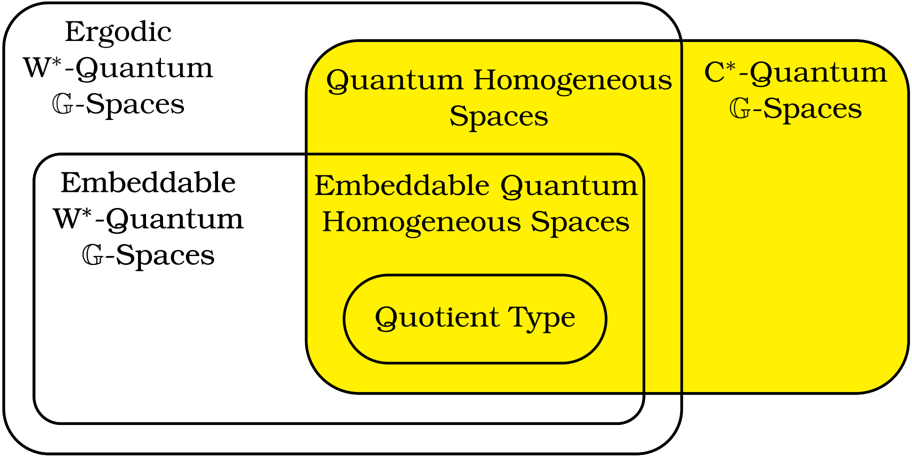

- C^*-algebra language, Podleś condition

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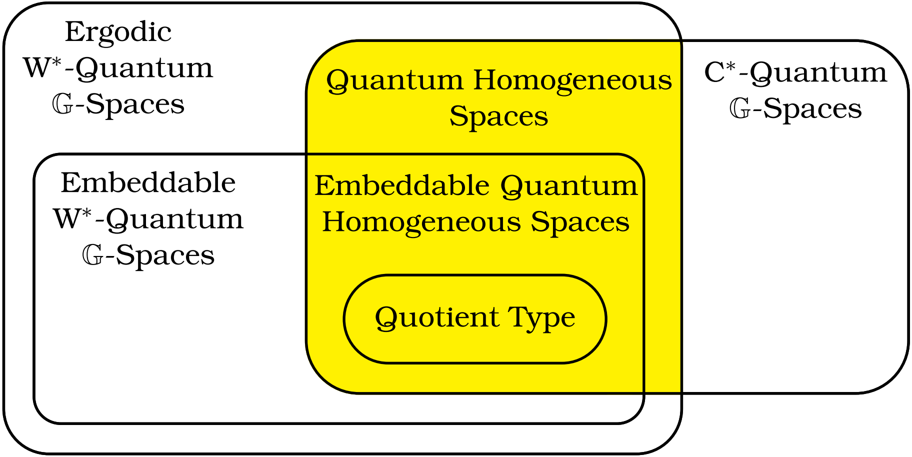

- Compatible C*- and von Neumann description

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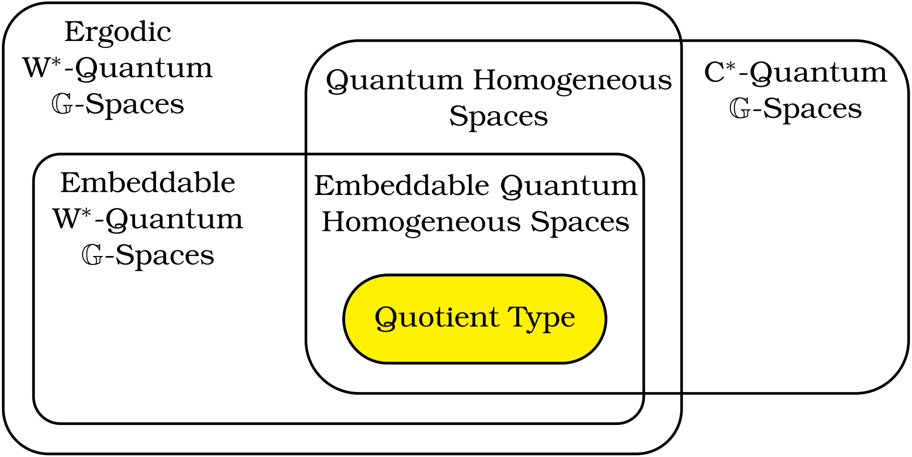

- Defined by S. Vaes, cf. work of P. Podleś

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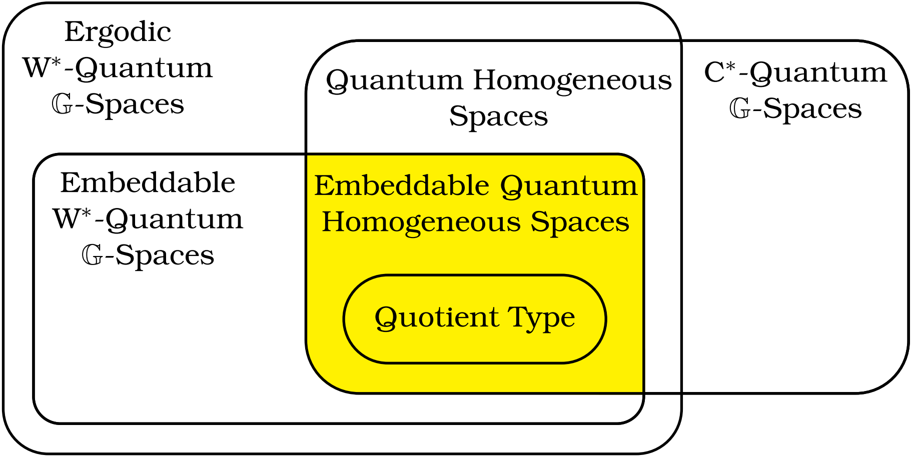

- Natural class we wish to study

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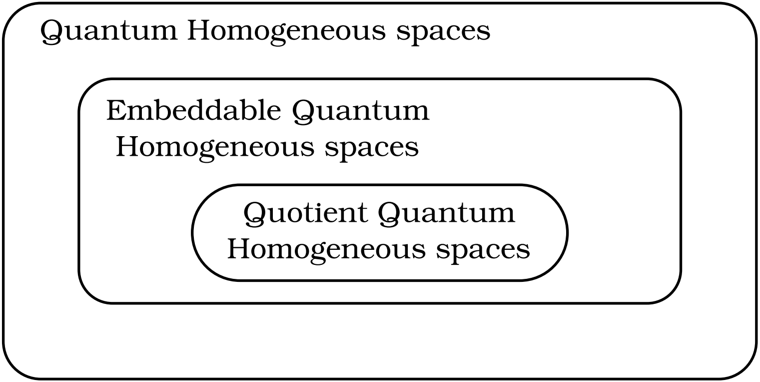

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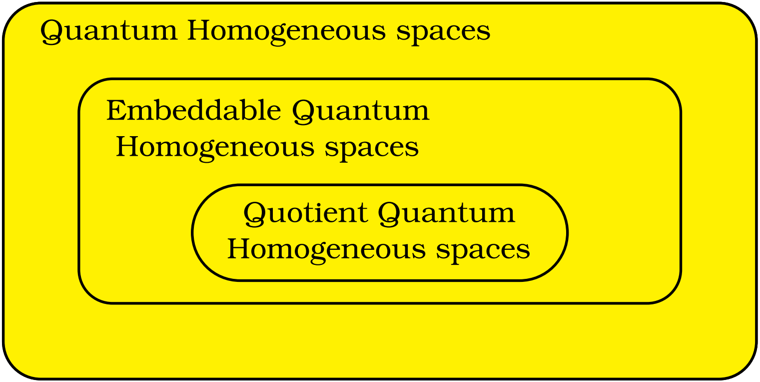

- Ergodic actions (transitivity)

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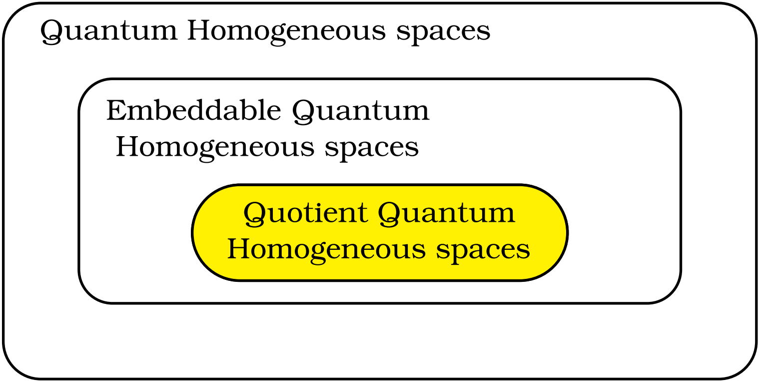

- Q.H.S.'s arising from subgroups (careful)

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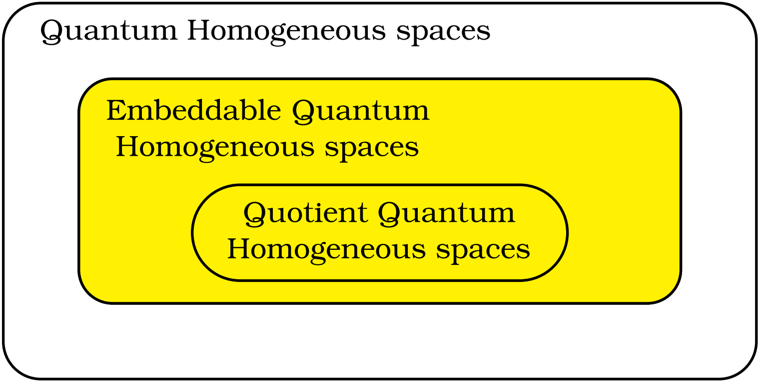

- Ergodic actions realized inside $C(\mathbb{G})$ via Δ

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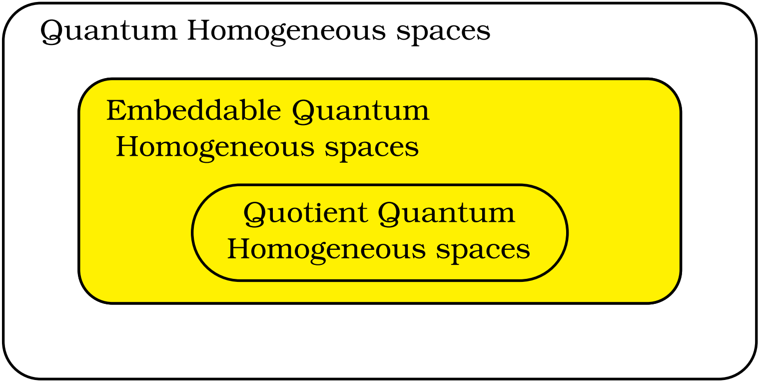

- Classically correspond to classical homogeneous spaces

Closed quantum subgroups

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- converse unclear, true in many cases.

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EMBEDDABLE W*-QUANTUM $\mathbb{G}$-SPACES

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Define $\widetilde{\mathbb{X}}$ by setting

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\end{aligned}$$

$\widetilde{\mathbb{X}}$ IS A W*-QUANTUM $\widehat{\mathbb{G}}$-SPACE

PROOF.

Take

- $y \in L^\infty(\widetilde{\mathbb{X}})$,
- $x \in L^\infty(\mathbb{X})$.

Then
$$\begin{aligned}
\Delta_{\widehat{\mathbb{G}}}(y)(\mathbb{1} \otimes x) &= \widehat{W}(y \otimes \mathbb{1})\widehat{W}^*(\mathbb{1} \otimes x) \\
&= \Sigma W^* \Sigma (y \otimes \mathbb{1}) \Sigma W \Sigma (\mathbb{1} \otimes x) \Sigma W^* W \Sigma \\
&= \Sigma W^* (\mathbb{1} \otimes y) W (x \otimes \mathbb{1}) W^* W \Sigma \\
&= \Sigma W^* (I \otimes y) \Delta_{\mathbb{G}}(x) W \Sigma \\
&= \Sigma W^* \Delta_{\mathbb{G}}(x)(\mathbb{1} \otimes y) W \Sigma \\
&= (\mathbb{1} \otimes x) \Sigma W^* (\mathbb{1} \otimes y) W \Sigma \\
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Theorem

$\mathbb{X}$ is of quotient type iff there exists a closed quantum subgroup $\mathbb{H}$ of $\mathbb{G}$ such that $L^\infty(\widetilde{\mathbb{X}})$ is the image of $L^\infty(\widehat{\mathbb{H}})$ in $L^\infty(\widehat{\mathbb{G}})$.

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is **strict**, i.e. strong*–strict continuous on $\|\cdot\|$-bounded subsets.

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- For classical groups, embeddable quantum homogeneous spaces correspond to homogeneous spaces.

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- In particular we find that **quantum** groups do not have diagonal subgroups.