Generation of C^*-algebras and the concept of a closed quantum subgroup

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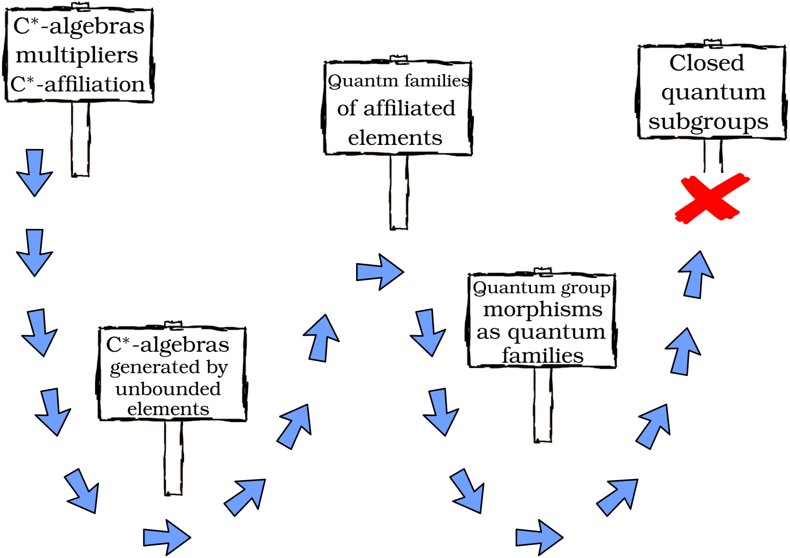
Talk outline



Multipliers and affiliated elements

Let A be a C*-algebra.

Definition

The **multiplier algebra** of A is

$$\mathrm{M}(A) = \mathcal{L}(A)$$

(adjointable operators on A as module over itself, $(a|b) = a^*b$).

Definition

Elements **affiliated** with A are closed $T\colon A \to A$ such that

- $\mathrm{pr}_1(\mathrm{Gr}(T))$ and $\mathrm{pr}_2(\mathrm{Gr}(T))$ are dense in A,
- $\mathrm{Gr}(T) \oplus \mathrm{Gr}(T)^\perp = A \oplus A$

(inner product: $\left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \middle| \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right) = a_1^* b_1 + a_2^* b_2$).

A^η — set of elements affiliated with A. $\mathrm{M}(A)$ is precisely the set of **bounded** elements of A^η.

Examples

A	$C_0(X)$	$\mathcal{K}(\mathscr{H})$	$C^*(G)$	$\bigoplus_\alpha M_{n_\alpha}(\mathbb{C})$ (c_0-sum)
$M(A)$	$C_b(X)$	$B(\mathscr{H})$	contains universal rep. $g \mapsto U_g$	$\bigoplus_\alpha M_{n_\alpha}(\mathbb{C})$ (ℓ^∞-sum)
A^η	$C(X)$	all closed operators on $\mathscr{H}$	$\mathfrak{g} \subset C^*(G)^\eta$ (Lie alg. of G)	$\prod_\alpha M_{n_\alpha}(\mathbb{C})$

Examples from quantum groups

Let

- $\mathbb{G}$ be a **compact quantum group** (described by $(\mathrm{C}(\mathbb{G}), \Delta)$),
- $(u^\alpha)_{\alpha\in\mathcal{R}}$ be a set of representatives of all equivalence classes of unitary representations of $\mathbb{G}$.

Then for each $\alpha \in \mathcal{R}$

- we have $u^\alpha \in M_{n_\alpha}(\mathbb{C}) \otimes \mathrm{C}(\mathbb{G})$,
- there is a strictly positive $F_\alpha \in M_{n_\alpha}(\mathbb{C})$ such that
    - $(\mathrm{id} \otimes \kappa^2)(u^\alpha) = (F_\alpha \otimes \mathbb{1})u(F_\alpha^{-1} \otimes \mathbb{1})$,
    - $\mathrm{Tr}\, F_\alpha = \mathrm{Tr}\, F_\alpha^{-1}$.

Consider the element $F = (F_\alpha)_{\alpha\in\mathcal{R}} \in \prod_{\alpha\in\mathcal{R}} M_{n_\alpha}(\mathbb{C})$.

Then $F \in c_0(\widehat{\mathbb{G}})^\eta$ (or $F \,\eta\, c_0(\widehat{\mathbb{G}})$), where

$$c_0(\widehat{\mathbb{G}}) = \bigoplus_{\alpha\in\mathcal{R}} M_{n_\alpha}(\mathbb{C}).$$

EXAMPLES FROM QUANTUM GROUPS CONTINUED

- We have the dual (discrete) quantum group $\widehat{\mathbb{G}}$ (described by $\bigl(c_0(\widehat{\mathbb{G}}),\widehat{\Delta}\bigr)$),
- $\widehat{\mathbb{G}}$ has left and right Haar measures $\widehat{h}_{\mathrm{L}}$, $\widehat{h}_{\mathrm{R}}$ (weights on $c_0(\widehat{\mathbb{G}})$).
- For $x \in \bigoplus_{\alpha\in\mathcal{R}} M_{n_\alpha}(\mathbb{C})$ (algebraic sum) we have

$$\begin{aligned}
(\mathrm{id}\otimes\widehat{h}_{\mathrm{L}})\widehat{\Delta}(x) &= \widehat{h}_{\mathrm{L}}(x)\mathbb{1},\\
(\widehat{h}_{\mathrm{R}}\otimes\mathrm{id})\widehat{\Delta}(x) &= \widehat{h}_{\mathrm{R}}(x)\mathbb{1},\\
(\widehat{h}_{\mathrm{L}}\otimes\mathrm{id})\widehat{\Delta}(x) &= \widehat{h}_{\mathrm{L}}(x)F^{2},\\
(\mathrm{id}\otimes\widehat{h}_{\mathrm{R}})\widehat{\Delta}(x) &= \widehat{h}_{\mathrm{R}}(x)F^{-2}.
\end{aligned}$$

Examples from quantum groups continued

- Non-compact quantum groups often come equipped with a finite dimensional representation (not unitary).
- The matrix elements of this representation are (unbounded) elements affiliated with $C_0(\mathbb{G})$.
- The quantum $E(2)$ group:
  - $\mathscr{H} = \ell^2(\mathbb{Z}^2)$ with o.n.b. $(e_{i,j})_{i,j\in\mathbb{Z}}$,
  - let $v\,e_{i,j} = e_{i-1,j}$, $n\,e_{i,j} = q^i e_{i,j+1}$, $(0 < q < 1)$,
  - $C_0(\mathbb{G})$ is the closure in $B(\mathscr{H})$ of

$$\left\{\sum f_k(n)v^k \,\middle|\, f_k \in C_0(\operatorname{Sp} n)\right\}.$$

  - We have:
    - $v \in M\bigl(C_0(\mathbb{G})\bigr)$,
    - $n \,\eta\, C_0(\mathbb{G})$,
    - $\left[\begin{smallmatrix} v & n \\ 0 & v^* \end{smallmatrix}\right]$ is the defining representation of $E_q(2)$.

Examples from quantum groups continued

- Consider the C^*-algebra B which is the closure of

$$\left\{\sum f_k(n)v^k\right\},$$

  where $f_k \in C_0(\operatorname{Sp} n)$ is such that $f(\mu z) = \mu^k f_k(z)$ for any $z \in \operatorname{Sp} n$ and $\mu \in \mathbb{T}$.
- Then $B \subset C_0(\mathbb{G})$ and $\alpha = \Delta\big|_B$ defines an action of $\mathbb{G}$ on B:
  - $\alpha \in \operatorname{Mor}\bigl(B, C_0(\mathbb{G}) \otimes B\bigr)$,
  - $(\mathrm{id} \otimes \alpha)\circ\alpha = (\Delta \otimes \mathrm{id})\circ\alpha$.
  - $\bigl[\bigl(C_0(\mathbb{G}) \otimes \mathbb{1}\bigr)\alpha(B)\bigr] = C_0(\mathbb{G}) \otimes B$.
- Moreover the operator vn is affiliated with B.

Morphisms of C*-algebras

Definition

Let A and C be C*-algebras. A **morphism** from A to C is a $*$-homomorphism

$$\Phi\colon A \longrightarrow \mathrm{M}(C)$$

such that $\Phi(A)C = C$.
$\mathrm{Mor}(A, C)$ denotes the set of all morphisms from A to C.

Proposition

Any $\Phi \in \mathrm{Mor}(A, C)$ *has a unique extension to a map* $A \,\reflectbox{$\eta$}\, T \longmapsto \Phi(T)\,\eta\,C$ *such that*

- $\mathrm{D}\big(\Phi(T)\big) = \Phi\big(\mathrm{D}(T)\big)$,
- $\Phi(T)\Phi(x) = \Phi(Tx)$ *for any* $x \in \mathrm{D}(T)$.

Generating C*-algebras by unbounded elements

Definition

Let A be a C*-algebra and $T_1, \ldots, T_n \,\eta\, A$. We say that A is *generated by* $T_1, \ldots, T_n \,\eta\, A$ if for

- any Hilbert space $\mathscr{H}$
- $\pi \in \mathrm{Mor}\big(A, \mathcal{K}(\mathscr{H})\big)$, (representation of A on $\mathscr{H}$)
- any non-degenerate C*-subalgebra $C \subset \mathrm{B}(\mathscr{H})$

we have

$$\Big[\pi(T_1), \ldots, \pi(T_n) \,\eta\, C\Big] \Longrightarrow \Big[\pi \in \mathrm{Mor}(A, C)\Big]$$

Comment: Given $C \subset \mathrm{B}(\mathscr{H})$ and a closed $S\colon \mathscr{H} \to \mathscr{H}$ the statement "$S \,\eta\, C$" makes sense:

- $\mathrm{D}(S) = \big\{c \in C \,\big|\, S \circ c \text{ extends to an element of } C\big\}$,
- $Sc =$ the extension of $S \circ c$.

Examples I

- The new notion of generation coincides with the usual one when $T_1, \dots, T_n \in A$.
- For $A = \mathrm{C}_0(X)$ a set of elements $T_1, \dots, T_n \,\eta\, A$ generates A iff
  - the functions $T_1, \dots, T_n \,\eta\, A$ separate points of X and
  - $\lim\limits_{x \to \infty} \sum\limits_{i=1}^{n} \bigl|T_i(x)\bigr|^2 = +\infty$

  (e.g. $T\colon \mathbb{R} \ni t \mapsto t$ generates $\mathrm{C}_0(\mathbb{R})$).
- The position and momentum operators $\boldsymbol{p}$ and $\boldsymbol{q}$ on $L^2(\mathbb{R})$ generate $\mathcal{K}\bigl(L^2(\mathbb{R})\bigr)$
  (cf. the Stone-von Neumann theorem).
- If G is a connected Lie group and $T_1, \dots, T_n$ is a basis of $\mathfrak{g}$ then $A = \mathrm{C}^*(G)$ is generated by $T_1, \dots, T_n \,\eta\, A$.

Examples II

- Let $\mathbb{G}$ be the quantum E(2) group (for some q). Then
$$\mathrm{C}_0(\mathbb{G}) = \overline{\left\{ \sum f_k(n) v^k \,\middle|\, f_k \in \mathrm{C}_0(\operatorname{Sp} n) \right\}}^{\|\cdot\|}$$
is generated by v and n (previously defined).
- Let
$$B = \overline{\left\{ \sum f_k(n) v^k \,\middle|\, f_k \text{ homogeneous of degree } k \right\}}^{\|\cdot\|}$$
be the C*-algebra with action of $\mathbb{G}$ (defined above). Then B is generated by vn.
- $B = \begin{bmatrix} \mathscr{T} & \mathcal{K} \\ \mathcal{K} & \mathcal{K} \end{bmatrix}$, where $\mathscr{T}$ is the Toeplitz algebra. Why isn't B commutative?

Families of affiliated elements

Let A be a C*-algebra. Then

- $\Big[T_1, \ldots, T_n \,\eta\, A\Big] \iff \left[T = \begin{bmatrix} T_1 \\ \vdots \\ T_n \end{bmatrix} \eta\ \mathbb{C}^n \otimes A\right]$

  (T corresponds to a family of affiliated elements indexed by the spectrum of the C*-algebra $\mathbb{C}^n$).

- If X is a locally compact space then

$$\big(\mathrm{C}_0(X) \otimes A\big)^\eta \cong \mathrm{C}(X, A^\eta)$$

  (with a natural topology on A^η).

- We have

$$\begin{bmatrix} v & n \\ 0 & v^* \end{bmatrix} \eta\ M_2(\mathbb{C}) \otimes \mathrm{C}_0\big(\mathrm{E}_q(2)\big).$$

C*-ALGEBRA GENERATED BY A QUANTUM FAMILY

Let A and D be C*-algebras.

- $T\,\eta\,D \otimes A$ is called a **quantum family** of elements affiliated with A indexed by (the spectrum of) D.

DEFINITION

We say that A is *generated by* $T\,\eta\,D \otimes A$ if for

- any Hilbert space $\mathscr{H}$
- $\pi \in \mathrm{Mor}\big(A, \mathcal{K}(\mathscr{H})\big)$,
- any non-degenerate C*-subalgebra $C \subset \mathrm{B}(\mathscr{H})$

we have

$$\Big[(\mathrm{id} \otimes \pi)(T)\,\eta\,D \otimes C\Big] \Longrightarrow \Big[\pi \in \mathrm{Mor}(A, C)\Big]$$

C*-algebras generated by quantum families

- Consider for each $s \in \mathbb{R}$ the function $e_s \colon \mathbb{R} \ni t \mapsto \mathrm{e}^{its}$.
- Then $E = (e_s)_{s \in \mathbb{R}}$ is a family of elements of $\mathrm{C_b}(X) = \mathrm{M}\big(\mathrm{C_0}(\mathbb{R})\big) \subset \mathrm{C_0}(\mathbb{R})^\eta$. Thus

$$E \in \mathrm{M}\big(\mathrm{C_0}(\mathbb{R}) \otimes \mathrm{C_0}(\mathbb{R})\big).$$

- $\mathrm{C_0}(\mathbb{R})$ is generated by the family $E \in \mathrm{M}\big(\mathrm{C_0}(\mathbb{R}) \otimes \mathrm{C_0}(\mathbb{R})\big)$.
- Let $\mathbb{G}$ be a locally compact quantum group and let

$$W \in \mathrm{M}\big(\mathrm{C_0}(\widehat{\mathbb{G}}) \otimes \mathrm{C_0}(\mathbb{G})\big)$$

  be the reduced bicharacter (multiplicative unitary).
- $\mathrm{C_0}(\mathbb{G})$ is generated by $W \in \mathrm{M}\big(\mathrm{C_0}(\widehat{\mathbb{G}}) \otimes \mathrm{C_0}(\mathbb{G})\big)$.

Homomorphisms of quantum groups

Let $\mathbb{G}$ and $\mathbb{H}$ be locally compact quantum groups. There is a one to one correspondence between

1. **strong quantum homomorphisms**: morphisms
$$\pi \in \operatorname{Mor}\bigl(\mathrm{C}_0^{\mathrm{u}}(\mathbb{G}), \mathrm{C}_0^{\mathrm{u}}(\mathbb{H})\bigr)$$
such that $(\pi \otimes \pi) \circ \Delta^{\mathrm{u}}_{\mathbb{G}} = \Delta^{\mathrm{u}}_{\mathbb{H}} \circ \pi$,
2. **bicharacters** (from $\mathbb{H}$ to $\mathbb{G}$): unitaries
$$V \in \mathrm{M}\bigl(\mathrm{C}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0(\mathbb{H})\bigr)$$
such that $$(\Delta_{\widehat{\mathbb{G}}} \otimes \mathrm{id}_{\mathrm{C}_0(\mathbb{H})})(V) = V_{23} V_{13},$$ $$(\mathrm{id}_{\mathrm{C}_0(\widehat{\mathbb{G}})} \otimes \Delta_{\mathbb{H}})(V) = V_{12} V_{13},$$
3. **right quantum homomorphisms**: morphisms
$$\rho \in \operatorname{Mor}\bigl(\mathrm{C}_0(\mathbb{G}), \mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\mathbb{H})\bigr)$$
such that $$(\Delta_{\mathbb{G}} \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \rho) \circ \Delta_{\mathbb{G}},$$ $$(\mathrm{id} \otimes \Delta_{\mathbb{H}}) \circ \rho = (\rho \otimes \mathrm{id}) \circ \rho.$$

Homomorphisms of quantum groups

$\mathbb{G}$, $\mathbb{H}$ — locally compact quantum groups.

- Strong quantum homomorphisms
$$\pi \in \mathrm{Mor}\big(\mathrm{C}_0^{\mathrm{u}}(\mathbb{G}), \mathrm{C}_0^{\mathrm{u}}(\mathbb{H})\big),$$
- bicharacters
$$V \in \mathrm{M}\big(\mathrm{C}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0(\mathbb{H})\big),$$
- right quantum homomorphisms
$$\rho \in \mathrm{Mor}\big(\mathrm{C}_0(\mathbb{G}), \mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\mathbb{H})\big)$$

(as defined above) describe homomorphisms $\mathbb{H} \to \mathbb{G}$.

Proposition

Any homomorphism from $\mathbb{H}$ to $\mathbb{G}$ defines uniquely a homomorphism $\widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$ via
$$\mathrm{M}\big(\mathrm{C}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0(\mathbb{H})\big) \ni V \longmapsto \widehat{V} = \sigma(V)^* \in \mathrm{M}\big(\mathrm{C}_0(\mathbb{H}) \otimes \mathrm{C}_0(\widehat{\mathbb{G}})\big).$$
This defines a transformation
$$\mathrm{Mor}\big(\mathrm{C}_0^{\mathrm{u}}(\mathbb{G}), \mathrm{C}_0^{\mathrm{u}}(\mathbb{H})\big) \ni \pi \longmapsto \widehat{\pi} \in \mathrm{Mor}\big(\mathrm{C}_0^{\mathrm{u}}(\widehat{\mathbb{H}}), \mathrm{C}_0^{\mathrm{u}}(\widehat{\mathbb{G}})\big)$$

Definition and its motivation

Definition (S.L. Woronowicz)

Let $\mathbb{G}$ and $\mathbb{H}$ be locally compact quantum groups. A homomorphism from $\mathbb{H}$ to $\mathbb{G}$ corresponding to a bicharacter $V \in \mathrm{M}\big(\mathrm{C}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0(\mathbb{H})\big)$ identifies $\mathbb{H}$ with **closed quantum subgroup** of $\mathbb{G}$ if

- $\mathrm{C}_0(\mathbb{H})$ is generated by $V \in \mathrm{M}\big(\mathrm{C}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0(\mathbb{H})\big)$.

Theorem

Let G and H be locally compact groups. Consider G and H as locally compact quantum groups. Then H is a closed quantum subgroup of G if and only if there is a (classical) homomorphism mapping H homeomorphically onto a closed subgroup of G.

First major result

Theorem

Let $\mathbb{G}$, $\mathbb{H}$ be locally compact quantum groups and consider a homomorphism from $\mathbb{H}$ to $\mathbb{G}$ described by

- *a bicharacter* $V \in \mathrm{M}\big(\mathrm{C}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0(\mathbb{H})\big)$,
- *a strong q. homomorphism* $\pi \in \mathrm{Mor}\big(\mathrm{C}_0^{\mathrm{u}}(\mathbb{G}), \mathrm{C}_0^{\mathrm{u}}(\mathbb{H})\big)$,
- *a right q. homomorphism* $\rho \in \mathrm{Mor}\big(\mathrm{C}_0(\mathbb{G}), \mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\mathbb{H})\big)$.

Then the following conditions are equivalent:

1. $\mathbb{H}$ *is a closed quantum subgroup of* $\mathbb{G}$ *(i.e.* $V \in \mathrm{M}\big(\mathrm{C}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0(\mathbb{H})\big)$ *generates* $\mathrm{C}_0(\mathbb{H})$*),*
2. *the right quantum homomorphism* ρ *satisfies*
$$\big[\rho\big(\mathrm{C}_0(\mathbb{G})\big)\big(\mathrm{C}_0(\mathbb{G}) \otimes \mathbb{1}_{\mathrm{C}_0(\mathbb{H})}\big)\big] = \mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\mathbb{H}),$$
3. $\pi\big(\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})\big) = \mathrm{C}_0^{\mathrm{u}}(\mathbb{H})$.

For duals of groups

Theorem

Let G and H be locally compact groups. Let π be a strong quantum homomorphism describing a homomorphism $\widehat{H} \to \widehat{G}$ and let $\widehat{\pi}$ correspond to the dual homomorphism $G \to H$, so that

$$\widehat{\pi}\colon\ \mathrm{C}_0(H) \ni f \longmapsto f \circ \theta \in \mathrm{M}\big(\mathrm{C}_0(G)\big)$$

for some continuous homomorphism $\theta\colon G \to H$. Then the following conditions are equivalent:

1. *$\widehat{H}$ is a closed quantum subgroup of $\widehat{G}$ (via the homomorphism corresponding to π);*
2. *θ maps G onto H and the induced map*

   $$\tilde{\theta}\colon\ G/_{\ker\theta} \longrightarrow H$$

   is a homeomorphism.

Another notion of quantum subgroup

- S. Vaes defined and used a notion of a closed quantum subgroup (c.q.s.) of a locally compact quantum group utilizing both dual homomorphism and von Neumann algebraic picture of L.C.Q.G.s.
- If $\mathbb{H}$ is a c.q.s. of $\mathbb{G}$ in the sense of Vaes then it is also a c.q.s. of $\mathbb{G}$ in the sense described above.
- For classical groups and duals of classical groups the two definitions of a c.q.s. are equivalent.
- If $\mathbb{H}$ is compact then both notions of a c.q.s. are equivalent.
- If $\mathbb{G}$ is discrete then both notions of a c.q.s. are equivalent.
- Let $\mathbb{H}$ be a c.q.s. of $\mathbb{G}$. Then $\mathbb{H}$ is a c.q.s. in the sense of Vaes if and only if the **quantum Herz restriction theorem** holds for the pair $(\mathbb{G}, \mathbb{H})$.
  (More details in the talk of P. Kasprzak.)