LECTURE III: ACTIONS OF COMPACT QUANTUM GROUPS

DEFINITION: (A,Δ) – COMPACT QUANTUM GROUP

B – UNITAL C^*-ALGEBRA

AN ACTION OF (A,Δ) ON B IS A UNITAL $*$-HOMOMORPHISM $\alpha: B \longrightarrow B\otimes A$ SUCH THAT

- $(\alpha\otimes \mathrm{id})\circ\alpha = (\mathrm{id}\otimes\Delta)\circ\alpha$,
- $\mathrm{span}\{\alpha(b)(\mathbb{1}\otimes a) \mid b\in B,\ a\in A\}$ IS DENSE IN $B\otimes A$.

CLASSICAL CASE: AN ACTION OF A COMPACT GROUP G ON A COMPACT SPACE X IS A CONTINUOUS MAP $X\times G \longrightarrow X$ (WRITTEN $(x,g)\longmapsto xg$) SUCH THAT

(i) $(xg_1)g_2 = x(g_1g_2)$ $\quad\forall x\in X,\ g_1,g_2\in G$

(ii) $xe = x$ $\quad\forall x\in X$

APPLICATION OF THE FUNCTOR $C(\cdot)$ YIELDS

$$\alpha: C(X) \longrightarrow C(X\times G) \cong C(X)\otimes C(G)$$

EXERCISE: (i) IS EQUIVALENT TO

$(\alpha\otimes \mathrm{id})\circ\alpha = (\mathrm{id}\otimes\Delta)\circ\alpha$

WHERE $\Delta: C(G) \longrightarrow C(G)\otimes C(G)$

IS THE STANDARD COMULTIPLICATION ON $C(G)$.

PROPOSITION: B – UNITAL C^*-ALGEBRA
G – COMPACT GROUP
$\alpha : B \to B \otimes C(G)$ UNITAL $*$-HOMOMORPHISM
$(\alpha \otimes id) \circ \alpha = (id \otimes \Delta) \circ \alpha$

THEN THE FOLLOWING ARE EQUIVALENT:

(1) THERE IS AN ACTION OF G ON B
(I.E. A HOMOMORPHISM $\tilde{\alpha} : G \to Aut(B)$ SUCH THAT $\forall\, b \in B$ THE MAP $G \ni t \mapsto \tilde{\alpha}_t(b) \in B$ IS CONTINUOUS)
SUCH THAT $(\alpha(b))(t) = \tilde{\alpha}_t(b)$ $\forall\, b \in B,\ t \in G$

(2) $\mathrm{span}\{\alpha(b)(1\!\!1 \otimes f) \mid b \in B,\ f \in C(G)\}$ IS DENSE IN $B \otimes C(G)$.

PROOF: $(1) \Rightarrow (2)$

WE IDENTIFY $B \otimes C(G)$ WITH $C(G, B)$. THE MAP

$$\Phi_0 : B \otimes_{alg} C(G) \ni b \otimes f \longmapsto \alpha(b)(1\!\!1 \otimes f) \in B \otimes C(G)$$

EXTENDS TO A UNITAL $*$-HOMOMORPHISM

$$\Phi : B \otimes C(G) \longrightarrow B \otimes C(G).$$

WE HAVE FOR $F \in B \otimes C(G) = C(G, B)$

$$(\Phi(F))(t) = \tilde{\alpha}_t(F(t)) \qquad (t \in G).$$

THUS FOR ANY $F \in C(G, B)$ WE HAVE $F = \Phi(\tilde{F})$, WHERE $\tilde{F}(t) = \tilde{\alpha}_{t^{-1}}(F(t))$. THIS MEAN THAT Φ IS ONTO. IT IS ALSO CLEARLY INJECTIVE, SO $\Phi \in Aut(C(G) \otimes B)$

IT FOLLOWS THAT

$$\mathrm{span}\{\alpha(b)(1\!\!1 \otimes f) \mid b \in B,\ f \in C(G)\}$$

IS DENSE IN $B \otimes C(G)$ AS THE IMAGE OF $B \otimes_{alg} C(G)$ UNDER AN AUTOMORPHISM.

(2) $\Rightarrow$ (1) DEFINE $\tilde{\alpha}_t = (id \otimes \delta_t) \circ \alpha : B \to B$

EVALUATION FUNCTIONAL $C(G) \ni f \mapsto f(t) \in \mathbb{C}$

THEN $\tilde{\alpha}_t \circ \tilde{\alpha}_s = \tilde{\alpha}_{ts}$ $\forall\, t,s \in G$. THIS IS AN ACTION OF G IFF THE IDEMPOTENT ENDOMORPHISM $\tilde{\alpha}_e : B \to B$ IS THE IDENTITY.

ASSUME THAT $\mathrm{span}\{\alpha(b)(1 \otimes f) \mid b \in B,\ f \in C(G)\}$ IS DENSE IN $B \otimes C(G)$. THEN FOR ANY $b \in B$ THE TENSOR $b \otimes 1$ CAN BE APPROXIMATED BY ELEMENTS FROM THIS SET, I.E.

$$b \otimes 1 = \lim_{n \to \infty} \sum_{i=1}^{N_n} \alpha(b_i^n)(1 \otimes f_i^n).$$

APPLYING $id \otimes \delta_e$ TO BOTH SIDES GIVES

$$b = \lim_{n \to \infty} \sum_{i=1}^{N_n} \tilde{\alpha}_e(b_i^n)\, f_i^n(e)$$

$$= \lim_{n \to \infty} \tilde{\alpha}_e \left(\sum_{i=1}^{N_n} f_i^n(e)\, b_i^n \right),$$

SO $\mathrm{Ran}\, \tilde{\alpha}_e = B$ AND $\tilde{\alpha}_e = id$. ∎

EXAMPLES OF ACTIONS:

(0) ACTION OF A COMPACT GROUP ON A UNITAL C^*-ALGEBRA (PARTICULAR CASE: ACTION ON A COMPACT SPACE)

(1) FIX $n \in \mathbb{N}$. $X := \{1, \dots, n\}$, $G := S_n$ PERMUTATION GROUP.

$X \times G \longrightarrow X$ — (ALMOST) STANDARD ACTION.

THIS GIVES RISE TO AN ACTION OF $(A = C(G), \Delta)$ ON $B = C(X) \cong \mathbb{C}^n$.

LET'S LOOK AT $G = S_n$ AS THE GROUP OF PERMUTATION MATRICES

$$\begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$(a_{ij})^2 = a_{ij} \quad (= \overline{a_{ij}}) \qquad \forall\, i,j$$

$$\sum_{j=1}^{n} a_{ij} = 1 \qquad \forall\, i$$

$$\sum_{i=1}^{n} a_{ij} = 1 \qquad \forall\, j$$

THUS

$$C(G) = C^*\left(a_{ij} \mid a_{ij} = (a_{ij})^2 = \overline{a_{ij}},\ \sum_{i=1}^{n} a_{ij} = 1\!\!1,\ \sum_{j=1}^{n} a_{ij} = 1\!\!1,\ [a_{ij}, a_{kl}] = 0\right)$$

SIMILARLY

$$C(X) = C^*\left(e_i \mid i = 1, \dots, n,\ e_i = e_i^* = e_i^2\ \forall i,\ \sum_{j=1}^{n} e_j = 1\!\!1\right)$$

THE ACTION IS DESCRIBED BY

$$\alpha_0 : C(X) \longrightarrow C(X) \otimes C(G)$$

$$\alpha_0(e_j) = \sum_{i=1}^{n} e_i \otimes a_{ij} \qquad (j = 1, \dots, n).$$

LET US CONSIDER A DIFFERENT C^*-ALGEBRA

$$A := C^*\left(a_{ij} \mid ij=1,\dots,n,\ a_{ij}=a_{ij}^2=a_{ij}^*,\ \sum_{i=1}^{n} a_{ij}=1,\ \sum_{j=1}^{n} a_{ij}=1\right)$$

WE HAVE A UNITAL $*$-HOMOMORPHISM

$$\alpha: B=C(X) \ni e_j \longmapsto \sum_{i=1}^{n} e_i \otimes a_{ij} \in B\otimes A.$$

<u>THEOREM (S. WANG):</u>

(1) THERE IS A UNIQUE $\Delta: A \to A\otimes A$ MAKING (A,Δ) A COMPACT QUANTUM GROUP SUCH THAT

$$\alpha: B \to B\otimes A$$

IS AN ACTION OF (A,Δ) ON B.

(2) IF (C,Δ_C) IS A COMPACT QUANTUM GROUP AND $\beta: B \to B\otimes C$ IS AN ACTION OF (C,Δ_C) ON B THEN THERE IS A UNIQUE $\Phi: B \to C$ SUCH THAT

$$\beta = (id\otimes\Phi)\circ\alpha.$$

MOREOVER $(\Phi\otimes\Phi)\circ\Delta = \Delta_C\circ\Phi$.

<u>REMARK:</u> POINT (2) ABOVE CHARACTERIZES THE COMPACT QUANTUM GROUP (A,Δ) UNIQUELY.

<u>THEOREM (S. WANG):</u> FOR $n=1,2,3$ $A\cong C(S_n)$. FOR $n\geq 4$ A IS NON-COMMUTATIVE AND $\dim A=\infty$.

THE COMPACT QUANTUM GROUPS (A,Δ) DISCUSSED ABOVE ARE CALLED THE <u>QUANTUM PERMUTATION GROUPS</u>.

<u>NOTE:</u> LET $\mu : B \longrightarrow \mathbb{C}$ BE THE FUNCTIONAL ("MEASURE") WHICH MAPS EACH e_i TO $\frac{1}{n}$. THEN

$$(\mu \otimes id)\,\alpha(b) = \mu(b)\,\mathbb{1} \qquad \forall\, b \in B$$

(<u>EXERCISE:</u> CHECK THIS).

THIS PROPERTY IS CALLED <u>INVARIANCE</u> OF μ WITH RESPECT TO α (μ IS <u>PRESERVED</u> BY α).

<u>(2)</u> LET US CONSIDER $B = M_2(\mathbb{C})$ AND FOR $q \in]0,1]$ LET ω_q BE THE FUNCTIONAL

$$B \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{a + q^2 d}{1 + q^2} \in \mathbb{C}$$

(THIS IS THE <u>POWERS</u> <u>STATE</u> ON M_2).

<u>PROPOSITION:</u> $B := M_2$, $(A, \Delta) := S_qU(2)$. THERE IS A UNIQUE MAP $\Psi_q : B \longrightarrow B \otimes A$ SUCH THAT

$$\Psi_q\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} -q\alpha\gamma & \alpha^2 \\ -q\gamma^2 & q^{-1}\alpha\gamma \end{pmatrix} \in M_2(A) = B \otimes A.$$

MOREOVER Ψ_q IS AN ACTION OF $S_qU(2)$ ON B PRESERVING THE POWERS STATE.

A FEW MORE DETAILS:

- THE 3-DIMENSIONAL IRREDUCIBLE REPRESENTATION OF $S_qU(2)$

$$w' = \begin{pmatrix} 1\!\!1 - (1+q^2)\gamma^*\gamma & \sqrt{1+q^2}\,\gamma\alpha & -\sqrt{1+q^2}\,\gamma^*\alpha^* \\ -\sqrt{1+q^2}\,\alpha\gamma^* & \alpha^2 & q^2\gamma^{*2} \\ \sqrt{1+q^2}\,\alpha^*\gamma & \gamma^2 & \alpha^{*2} \end{pmatrix}$$

- LET C BE THE C^*-SUBALGEBRA OF A GENERATED BY MATRIX ELEMENTS OF w'. THEN $\Delta(C) \subset C \otimes C$ AND WITH $\Delta_C = \Delta|_C$ THE PAIR (C, Δ_C) IS A COMPACT QUANTUM GROUP.
  THIS IS THE QUANTUM SO(3) GROUP $(S_qO(3))$.

- THE ACTION ψ_q OF $S_qU(2)$ ON $B = M_2$ IS REALLY AN ACTION OF $S_qO(3)$, I.E. $\psi_q(B) \subset B \otimes C$.

THEOREM: LET (D, Δ_D) BE A COMPACT QUANTUM GROUP AND LET $\varphi: M_2 \to M_2 \otimes D$ BE AN ACTION OF (D, Δ_D) ON M_2 PRESERVING THE POWERS STATE ω_q. THEN THERE EXISTS A UNIQUE UNITAL $*$-HOMOMORPHISM $\Phi: C \to D$ SUCH THAT

$$\varphi = (\mathrm{id} \otimes \Phi) \circ \psi_q.$$

FACT: LET (D, Δ_D) BE A COMPACT QUANTUM GROUP ACTING ON A FINITE-DIMENSIONAL C^*-ALGEBRA N. THEN THERE EXISTS A FAITHFUL STATE ON N INVARIANT FOR THE ACTION OF (D, Δ_D).

THEOREM: LET (D, Δ_D) BE A COMPACT QUANTUM GROUP ACTING ON M_2:

$$\varphi : M_2 \longrightarrow M_2 \otimes D$$

THEN THERE EXIST

- $q \in]0,1]$
- UNITARY $u \in M_2$
- $\Phi : C \longrightarrow D$, WHERE $(C, \Delta_C) = S_qO(3)$

SUCH THAT

$$\varphi(m) = (id \otimes \Phi)\Big((u \otimes 1\!\!1)\, \Psi_q(u^* m u)\, (u^* \otimes 1\!\!1)\Big) \qquad \forall\, m \in M_2 .$$