LECTURE II: COMPACT QUANTUM GROUPS AND REPRESENTATION THEORY

DEFINITION: A COMPACT QUANTUM GROUP IS A PAIR (A, Δ) SUCH THAT

A IS A UNITAL C^*-ALGEBRA

$\Delta : A \to A \otimes A$ IS A UNITAL $*$-HOMOMORPHISM

AND

- $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$
- $\left.\begin{array}{l} \mathrm{span}\{\Delta(a)(1 \otimes b) \mid a, b \in A\} \\ \mathrm{span}\{(a \otimes 1)\Delta(b) \mid a, b \in A\} \end{array}\right\}$ ARE DENSE IN $A \otimes A$.

EXAMPLES: (1) $S_qU(2)$

(2) G – COMPACT GROUP

$A := C(G)$

$\Delta : C(G) \to C(G) \otimes C(G) \cong C(G \times G)$

$f \in C(G)$, $\Delta(f) \in C(G \times G)$

$$(\Delta(f))(x, y) = f(xy)$$

EXERCISE: CHECK THAT

$\left.\begin{array}{l} \mathrm{span}\{\Delta(f)(1 \otimes g) \mid f, g \in C(G)\} \\ \mathrm{span}\{(f \otimes 1)\Delta(g) \mid f, g \in C(G)\} \end{array}\right\}$ ARE DENSE IN $C(G \times G)$

(3) Γ – DISCRETE GROUP, $A := C^*(\Gamma)$

$\Gamma \subset \mathbb{C}\Gamma \subset A$, $\exists!\ \Delta : A \to A \otimes A$

$$\Delta(\gamma) = \gamma \otimes \gamma \quad (\gamma \in \Gamma).$$

FACT: LET (A, Δ) BE A COMPACT QUANTUM GROUP WITH A - COMMUTATIVE. THEN THERE IS A COMPACT SPACE G AND A CONTINUOUS $\mu : G \times G \longrightarrow G$ SUCH THAT

$$A \cong C(G)$$

$$(\Delta(f))(x, y) = f(\mu(x, y))$$

AND (EXERCISE)

- $\mu(x, \mu(y, z)) = \mu(\mu(x, y), z) \quad \forall\, x, y, z \in G$
- FOR $x, y \in G$

$$[\exists z \;\; \mu(x, z) = \mu(y, z)] \Rightarrow [x = y],$$

$$[\exists z \;\; \mu(z, x) = \mu(z, y)] \Rightarrow [x = y].$$

EXERCISE: A COMPACT TOPOLOGICAL SEMIGROUP WITH CANCELLATION PROPERTIES IS A TOPOLOGICAL GROUP.

THEOREM (S.L. WORONOWICZ, A. VAN DAELE):
LET (A, Δ) BE A COMPACT QUANTUM GROUP. THEN THERE EXISTS A UNIQUE STATE h ON A SUCH THAT

$$(\mathrm{id} \otimes h)\Delta(a) = h(a)\mathbb{1} = (h \otimes \mathrm{id})\Delta(a) \qquad \forall\, a \in A.$$

DEFINITION: THIS STATE h IS THE <u>HAAR MEASURE</u> OF (A, Δ).

- IF $A = C(G)$ THEN $h(f) = \int_G f(x)\,dx$ (HAAR MEASURE)

- IF $A = C^*(\Gamma)$ THEN $h(a) = (\delta_e \mid \lambda(a)\, \delta_e)$, WHERE $\lambda : C^*(\Gamma) \longrightarrow B(\ell^2(\Gamma))$ IS THE REGULAR REPRESENTATION.

DEFINITION: LET (A, Δ) BE A COMPACT QUANTUM GROUP. A FINITE-DIMENSIONAL REPRESENTATION OF (A, Δ) IS AN INVERTIBLE MATRIX $v \in M_n \otimes A$ SUCH THAT $(id_{M_n} \otimes \Delta)\, v = v_1 v_2$

EXERCISE: $v \in M_n \otimes A$, $v = \begin{pmatrix} v_{11} & \dots & v_{1n} \\ \vdots & & \vdots \\ v_{n1} & \dots & v_{nn} \end{pmatrix}$ IS A REPRESENTATION OF (A, Δ) IFF

- v IS INVERTIBLE

AND

- $$\Delta(v_{kl}) = \sum_{p=1}^{n} v_{kp} \otimes v_{pl}, \qquad k, l = 1, \dots, n.$$

- $M_n \otimes A$ IS A $*$-ALGEBRA (C^*-ALGEBRA)
- WE SAY THAT A REPRESENTATION $v \in M_n \otimes A$ IS UNITARY IF $v^* v = v v^* = 1\!\!1$ $(= 1\!\!1_{M_n} \otimes 1\!\!1_A)$.

FACT: IF $A = C(G)$ THEN $v \in M_n \otimes A \cong C(G, M_n)$ IS A REPRESENTATION OF (A, Δ) IFF

$$v \in C(G, M_n)$$

HAS RANGE IN THE SET OF INVERTIBLE MATRICES AND $v(x) v(y) = v(xy)$ FOR ALL $x, y \in G$.

FACT: IF $A = C^*(\Gamma)$ AND $v \in M_n \otimes A$ IS A REPRESENTATION OF (A, Δ) THEN THERE IS AN INVERTIBLE MATRIX $c \in M_n(\mathbb{C})$ AND $\gamma_1, \dots, \gamma_n \in \Gamma$ SUCH THAT

$$v = c \begin{pmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{pmatrix} c^{-1}$$

<u>THEOREM:</u> LET (A,Δ) BE A COMPACT QUANTUM GROUP AND LET $v \in M_n \otimes A$ BE A REPRESENTATION OF (A,Δ). THEN THERE EXISTS AN INVERTIBLE MATRIX $T \in M_n(\mathbb{C})$ SUCH THAT

$$(T \otimes 1\!\!1)\, v\, (T^{-1} \otimes 1\!\!1)$$

IS A UNITARY REPRESENTATION OF (A,Δ).

<u>PROOF:</u> $v \in M_n \otimes A$ IS INVERTIBLE, SO v^*v IS POSITIVE AND INVERTIBLE. THEREFORE $\mathrm{Sp}\, v^*v$ IS SEPARATED FROM 0. THUS $v^*v \geq \delta 1\!\!1_{M_n \otimes A}$ FOR SOME $\delta > 0$.

LET $Q = (id \otimes h)(v^*v) \in M_n(\mathbb{C})$. h IS POSITIVE, SO

$$Q \geq \delta (id \otimes h) 1\!\!1 = \delta 1\!\!1_{M_n}.$$

IT FOLLOWS THAT Q IS INVERTIBLE. LET $T = Q^{1/2}$.

PUT $w = (T \otimes 1\!\!1)\, v\, (T^{-1} \otimes 1\!\!1)$.

LET US CHECK THAT w IS UNITARY:

$$w^*w = (T^{-1} \otimes 1\!\!1)\, v^* (Q \otimes 1\!\!1)\, v\, (T^{-1} \otimes 1\!\!1)$$

NOW

$$\begin{aligned}
Q \otimes 1\!\!1 &= \left[(id \otimes h)(v^*v)\right] \otimes 1\!\!1 = (id \otimes h \otimes id)(id \otimes \Delta)(v^*v) \\
&= (id \otimes h \otimes id)\left((v_1 v_2)^* v_1 v_2\right) \\
&= (id \otimes h \otimes id)\left(v_2^* v_1^* v_1 v_2\right) \\
&= v^* \left[(id \otimes h \otimes h)(v_1^* v_1)\right] v \\
&= v^* \left[\left((id \otimes h)(v^*v)\right) \otimes 1\!\!1\right] v = v^* (Q \otimes 1\!\!1)\, v
\end{aligned}$$

SO THAT

$$w^*w = (T^{-1} \otimes 1\!\!1)(Q \otimes 1\!\!1)(T^{-1} \otimes 1\!\!1) = 1\!\!1$$

SIMILARLY $\quad ww^* = (T\otimes 1\!\!1)\, v\, (Q^{-1}\otimes 1\!\!1)\, v^*\, (T\otimes 1\!\!1)$.

WE KNOW ALREADY THAT $\quad Q\otimes 1\!\!1 = v^*(Q\otimes 1\!\!1)\, v$, SO

$$(v^*)^{-1}(Q\otimes 1\!\!1)\, v^{-1} = Q\otimes 1\!\!1.$$

TAKING INVERSES GIVES

$$v(Q^{-1}\otimes 1\!\!1)\, v^* = Q^{-1}\otimes 1\!\!1$$

AND SO

$$w w^* = (T\otimes 1\!\!1)\,(Q^{-1}\otimes 1\!\!1)(T\otimes 1\!\!1) = 1\!\!1.$$

THE LAST THING TO CHECK IS

$$\begin{aligned}
(id\otimes\Delta)w &= (id\otimes\Delta)\big((T\otimes 1\!\!1)\, v\, (T^{-1}\otimes 1\!\!1)\big)\\
&= (T\otimes 1\!\!1\otimes 1\!\!1)\; v_1 v_2\, (T^{-1}\otimes 1\!\!1\otimes 1\!\!1)\\
&= (T\otimes 1\!\!1\otimes 1\!\!1)\, v_1\, (T^{-1}\otimes 1\!\!1\otimes 1\!\!1)\;(T\otimes 1\!\!1\otimes 1\!\!1)\, v_2\, (T^{-1}\otimes 1\!\!1\otimes 1\!\!1)\\
&= \big[(T\otimes 1\!\!1)v(T^{-1}\otimes 1\!\!1)\big]_1\;\big[(T\otimes 1\!\!1)v(T^{-1}\otimes 1\!\!1)\big]_2 = w_1 w_2 .
\end{aligned}$$

$\blacksquare$

- LET $v \in M_n \otimes A$ AND $w \in M_m \otimes A$ BE REPRESENTATIONS OF (A, Δ). THE DIRECT SUM OF v AND w IS
$$v \oplus w = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \in M_{n+m} \otimes A$$

$v \oplus w$ IS A REPRESENTATION OF (A, Δ) AND IF v AND w ARE UNITARY THEN SO IS $v \oplus w$.
LET $P = \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & 0 \end{pmatrix}$. THEN P IS A PROJECTION AND
$$(P \otimes \mathbb{1})(v \oplus w) = (v \oplus w)(P \otimes \mathbb{1}).$$

- IF $u \in M_k \otimes A$ IS A UNITARY REPRESENTATION OF (A, Δ) AND $P \in M_k(\mathbb{C})$ IS A PROJECTION SUCH THAT
$$(P \otimes \mathbb{1}) u = u (P \otimes \mathbb{1})$$
THEN u IS EQUIVALENT TO A DIRECT SUM $u \sim v \oplus w$ OF TWO UNITARY REPRESENTATIONS

[EQUIVALENCE OF REPRESENTATIONS:
$$(u \sim w) \iff \left(\begin{array}{l} \exists \text{ INVERTIBLE } T \\ \text{SUCH THAT } (T \otimes \mathbb{1}) u = w (T \otimes \mathbb{1}) \end{array} \right)$$

DEFINITION: A REPRESENTATION $u \in M_n \otimes A$ OF (A, Δ) IS IRREDUCIBLE IF
$$\left(\begin{array}{c} P \in \mathrm{Proj}(M_n(\mathbb{C})) \\ (P \otimes \mathbb{1}) u = u (P \otimes \mathbb{1}) \end{array} \right) \Longrightarrow \left(P = 0 \vee P = \mathbb{1} \right).$$

THEOREM (S.L. WORONOWICZ):

ANY REPRESENTATION OF A COMPACT QUANTUM GROUP IS EQUIVALENT TO A DIRECT SUM OF IRREDUCIBLE REPRESENTATIONS.

REMARK: THE ABOVE THEOREM IS TRUE ALSO FOR INFINITE-DIMENSIONAL REPRESENTATIONS (UNITARY, STRONGLY CONTINUOUS) OF COMPACT QUANTUM GROUPS:

- (A,Δ) – COMPACT QUANTUM GROUP
- H – HILBERT SPACE
- $U \in M(\mathcal{K}(H) \otimes A)$ UNITARY

$$(\mathrm{id}\otimes\Delta)U = U_1 U_2$$

$$U_1 = U \otimes \mathbb{1} \in M(\mathcal{K}(H)\otimes A \otimes A)$$

$$U_2 = \Phi(U), \quad \Phi \in \mathrm{Mor}(\mathcal{K}(H)\otimes A,\ \mathcal{K}(H)\otimes A\otimes A),$$

$$\Phi(m\otimes a) = m \otimes \mathbb{1} \otimes a.$$

FACT: ANY IRREDUCIBLE REPRESENTATION OF A COMPACT QUANTUM GROUP IS FINITE-DIMENSIONAL.

THEOREM (S.L. WORONOWICZ):

LET (A,Δ) BE A COMPACT QUANTUM GROUP AND LET $\mathcal{A}$ BE THE SPAN OF MATRIX ELEMENTS OF ALL IRREDUCIBLE REPRESENTATIONS OF (A,Δ). THEN $\mathcal{A}$ IS A DENSE UNITAL $*$-SUBALGEBRA OF A. MOREOVER $\Delta(\mathcal{A}) \subset \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A}$ AND $(\mathcal{A}, \Delta|_{\mathcal{A}})$ IS A HOPF $*$-ALGEBRA.

- THE PROOF OF THE LAST THEOREM IS BASED ON THE NOTION OF THE REGULAR REPRESENTATION.

THIS CONSTRUCTION LIES OUTSIDE THE SCOPE OF THESE NOTES. LET US ONLY MENTION ITS SIMPLIFIED VERSION

ASSUME THAT THE HAAR MEASURE h OF (A, Δ) IS FAITHFUL. THEN A EMBEDDS AS A DENSE SUBSET OF THE GNS HILBERT SPACE H FOR h. ALSO $A \subset B(H)$. ONE CAN PROVE THAT THE MAPPING

$$A \otimes_{alg} A \ni a \otimes b \longmapsto \Delta(a)(1\!\!1 \otimes b) \in A \otimes A$$

EXTENDS TO A UNITARY $U \in B(H \otimes H)$. MOREOVER $U \in M(\mathcal{K}(H) \otimes A)$ AND

$$(\mathrm{id} \otimes \Delta) U = U_1 U_2 .$$

REPRESENTATIONS OF $S_q U(2)$:

- $s \in \frac{1}{2}\mathbb{N}$, $\quad T_s := \{-s, -s+1, \dots, s\}$
- $w^s := \left(w^s_{a,b}\right)_{a,b \in T_s}$ DEFINED BY

$$\Delta\left(\alpha^{s+k}(\gamma^*)^{s-k}\right) = \sum_{i \in T_s} \alpha^{s+i}(\gamma^*)^{s-i} \otimes w^s_{i,k}$$

- $w^0 := 1\!\!1$
- $\{w^s\}_{s \in \frac{1}{2}\mathbb{Z}_+}$ IS A COMPLETE LIST OF IRREDUCIBLE REPRESENTATIONS OF $S_q U(2)$ UP TO EQUIVALENCE.