# LECTURE I: QUANTUM SU(2)

- $q \in [-1,1] \setminus \{0\}$
- FOR $H$ – HILBERT SPACE AND $\dot\alpha, \dot\gamma \in B(H)$ WE SAY THAT $(\dot\alpha, \dot\gamma)$ <u>SATISFY</u> <u>$S_qU(2)$</u> <u>RELATIONS</u> IF

$$\dot\alpha\dot\gamma = q\dot\gamma\dot\alpha$$
$$\dot\gamma^*\dot\gamma = \dot\gamma\dot\gamma^* \qquad (\dot\alpha\dot\gamma = q\dot\gamma\dot\alpha)$$
$$\dot\alpha^*\dot\alpha + \dot\gamma^*\dot\gamma = \mathbb{1}$$
$$\dot\alpha\dot\alpha^* + q^2\dot\gamma^*\dot\gamma = \mathbb{1}$$

- $\mathcal{F}$ := FREE $*$-ALGEBRA GENERATED BY SYMBOLS $\alpha, \gamma, \mathbb{1}$

UNIT

- FOR $a \in \mathcal{F}$ WE LET

$$\|a\| := \sup \|\pi(a)\|$$

OVER ALL $*$-HOMOMORPHISMS $\pi : \mathcal{F} \to B(H)$
($H$ – HILBERT SPACE) SUCH THAT
$(\pi(\alpha), \pi(\gamma))$ SATISFY $S_qU(2)$ RELATIONS

<u>FACT:</u> $\|\cdot\|$ IS A $C^*$-SEMINORM ON $\mathcal{F}$

<u>EXERCISE:</u> $(\dot\alpha, \dot\gamma)$ SATISFY $S_qU(2)$ RELATIONS IFF

$$\begin{pmatrix} \dot\alpha & -q\dot\gamma^* \\ \dot\gamma & \dot\alpha^* \end{pmatrix} \in M_2 \otimes B(H) \cong B(H \oplus H)$$

IS UNITARY.

<u>EXERCISE:</u> FOR ANY $a \in \mathcal{F}$ $\|a\|$ IS FINITE.

- $A :=$ COMPLETION OF $\mathcal{F}/_{\{a \mid \|a\|=0\}}$

- FOR ANY PAIR $(\dot{\alpha}, \dot{\gamma})$ OF OPERATORS ON A HILBERT SPACE $H$ SUCH THAT $(\dot{\alpha}, \dot{\gamma})$ SATISFY $S_qU(2)$ RELATIONS THERE IS A UNIQUE $\pi \in \mathrm{Rep}(A, H)$ SUCH THAT

$$\pi(\alpha) = \dot{\alpha}, \quad \pi(\gamma) = \dot{\gamma}.$$

- EXAMPLE OF A REPRESENTATION OF $A$:
  - $H$ - HILBERT SPACE WITH ONB $(e_{n,k})_{\substack{n \in \mathbb{Z}_+ \\ k \in \mathbb{Z}}}$
  - $$\pi(\alpha)\, e_{n,k} = \sqrt{1-q^{2n}}\; e_{n-1,k}$$
    $$\pi(\gamma)\, e_{n,k} = q^n\, e_{n,k+1}$$

THIS REPRESENTATION IS FAITHFUL, SO WE MAY REGARD $A$ AS EMBEDDED INTO $B(H)$.

- MATRICES

$$\begin{pmatrix} \alpha\otimes 1\!\!1 & -q\gamma^*\otimes 1\!\!1 \\ \gamma\otimes 1\!\!1 & \alpha^*\otimes 1\!\!1 \end{pmatrix} \quad \text{AND} \quad \begin{pmatrix} 1\!\!1\otimes\alpha & -q 1\!\!1\otimes\gamma^* \\ 1\!\!1\otimes\gamma & 1\!\!1\otimes\alpha^* \end{pmatrix}$$

ARE UNITARY, SO THEIR PRODUCT

$$\begin{pmatrix} \alpha\otimes\alpha - q\gamma^*\otimes\gamma & -q\gamma^*\otimes\alpha^* - q\alpha\otimes\gamma^* \\ \gamma\otimes\alpha + \alpha^*\otimes\gamma & \alpha^*\otimes\alpha^* - q\gamma\otimes\gamma^* \end{pmatrix}$$

IS UNITARY.

THEREFORE $\exists!$ UNITAL $*$- HOMOMORPHISM

$$\Delta : A \longrightarrow A\otimes A$$

SUCH THAT

$$\Delta(\alpha) = \alpha\otimes\alpha - q\gamma^*\otimes\gamma$$

$$\Delta(\gamma) = \gamma\otimes\alpha + \alpha^*\otimes\gamma$$

- NOTICE THAT FOR $u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in M_2(A) \cong M_2\otimes A$

WE HAVE

$$(id_{M_2}\otimes\Delta)\, u = u_1 u_2,$$

WHERE

$$u_1 = \begin{pmatrix} \alpha\otimes 1\!\!1 & -q\gamma^*\otimes 1\!\!1 \\ \gamma\otimes 1\!\!1 & \alpha^*\otimes 1\!\!1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1\!\!1\otimes\alpha & -q 1\!\!1\otimes\gamma^* \\ 1\!\!1\otimes\gamma & 1\!\!1\otimes\alpha^* \end{pmatrix}.$$

<u>EXERCISE:</u> $(id_A\otimes\Delta)\circ\Delta = (\Delta\otimes id_A)\circ\Delta$

- ASSUME WE HAVE $v\in M_n\otimes A$ S.T. $v^*v = vv^* = 1\!\!1$ AND

$$(id_{M_n}\otimes\Delta)v = v_1 v_2,$$

WHERE

$$v_1 = \begin{pmatrix} v_{11}\otimes 1\!\!1 & \cdots & v_{1n}\otimes 1\!\!1 \\ \vdots & & \vdots \\ v_{n1}\otimes 1\!\!1 & \cdots & v_{nn}\otimes 1\!\!1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1\!\!1\otimes v_{11} & \cdots & 1\!\!1\otimes v_{1n} \\ \vdots & & \vdots \\ 1\!\!1\otimes v_{n1} & \cdots & 1\!\!1\otimes v_{nn} \end{pmatrix}$$

- WE HAVE $\big((id \otimes \Delta)v\big)v_2^* = v_1$ I.E.

$$\begin{pmatrix} \Delta(v_{11}) & \cdots & \Delta(v_{1n}) \\ \vdots & & \vdots \\ \Delta(v_{n1}) & \cdots & \Delta(v_{nn}) \end{pmatrix} \begin{pmatrix} 1\!\!1 \otimes v_{11}^* & \cdots & 1\!\!1 \otimes v_{n1}^* \\ \vdots & & \vdots \\ 1\!\!1 \otimes v_{1n}^* & \cdots & 1\!\!1 \otimes v_{nn}^* \end{pmatrix} = \begin{pmatrix} v_{11} \otimes 1\!\!1 & \cdots & v_{1n} \otimes 1\!\!1 \\ \vdots & & \vdots \\ v_{n1} \otimes 1\!\!1 & \cdots & v_{nn} \otimes 1\!\!1 \end{pmatrix}$$

OR

$$v_{kl} \otimes 1\!\!1 = \sum_{p=1}^{n} \Delta(v_{kp})(1\!\!1 \otimes v_{lp}^*)$$

QUESTION: ARE THERE MANY SUCH MATRICES $v$ ?

- TAKE $\left.\begin{array}{l} v \in M_n \otimes A \\ w \in M_m \otimes A \end{array}\right\}$ UNITARY

WITH

$$(id_{M_n} \otimes \Delta)v = v_1 v_2, \quad (id_{M_m} \otimes \Delta)w = w_1 w_2.$$

$$v = \sum_k m_k \otimes a_k, \quad v_1 = \sum_k m_k \otimes a_k \otimes 1\!\!1, \quad v_2 = \sum_k m_k \otimes 1\!\!1 \otimes a_k$$

$$w = \sum_l n_l \otimes b_l, \quad w_1 = \sum_l n_l \otimes b_l \otimes 1\!\!1, \quad w_2 = \sum_l n_l \otimes 1\!\!1 \otimes b_l$$

DEFINE

$$v \oplus w = \sum_{k,l} m_k \otimes n_l \otimes a_k b_l \in M_n \otimes M_m \otimes A \cong M_{n\cdot m} \otimes A$$

NOTE THAT $v \oplus w$ IS UNITARY BECAUSE

$$v \oplus w = \Big(\sum_k m_k \otimes 1\!\!1 \otimes a_k\Big)\Big(\sum_l 1\!\!1 \otimes n_l \otimes b_l\Big)$$

UNITARY

EXERCISE: PROVE THAT

$$(id_{M_{n\cdot m}} \otimes \Delta)(v \oplus w) = (v \oplus w)_1 (v \oplus w)_2.$$

- TAKE
$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in M_2 \otimes A$$

AND PUT

$$u \mathbin{\textcircled{\scriptsize T}} u = u^{\textcircled{\scriptsize T} 2} = \begin{pmatrix} \alpha\alpha & -q\alpha\gamma^* & -q\gamma^*\alpha & q^2\gamma^*\gamma^* \\ \alpha\gamma & \alpha\alpha^* & -q\gamma^*\gamma & -q\gamma^*\alpha^* \\ \gamma\alpha & -q\gamma\gamma^* & \alpha^*\alpha & -q\alpha^*\gamma^* \\ \gamma\gamma & \gamma\alpha^* & \alpha^*\gamma & \alpha^*\alpha^* \end{pmatrix} \in M_4 \otimes A$$

$$u^{\textcircled{\scriptsize T} 3} = u \mathbin{\textcircled{\scriptsize T}} u \mathbin{\textcircled{\scriptsize T}} u = \begin{pmatrix} \alpha\, u^{\textcircled{\scriptsize T} 2} & -q\gamma^* u^{\textcircled{\scriptsize T} 2} \\ \gamma\, u^{\textcircled{\scriptsize T} 2} & \alpha^* u^{\textcircled{\scriptsize T} 2} \end{pmatrix} \in M_8 \otimes A$$

$$\vdots$$

ANY MONOMIAL IN $\alpha, \gamma, \alpha^*, \gamma^*$ IS A MATRIX ELEMENT OF SOME $u^{\textcircled{\scriptsize T} n}$.

<u>COROLLARY:</u> $\mathrm{span}\{\Delta(a)(1\!\!1 \otimes b) \mid a, b \in A\}$ IS DENSE IN $A \otimes A$.

<u>PROOF:</u> RECALL THAT IF $v \in M_n \otimes A$ IS UNITARY AND

$$(\mathrm{id}_{M_n} \otimes \Delta) v = v_1 v_2$$

THEN $v_{kl} \otimes 1\!\!1 \in \mathrm{span}\{\Delta(a)(1\!\!1 \otimes b) \mid a, b \in A\}$,

SO $\forall\, c \in A$

$$v_{kl} \otimes c \in \mathrm{span}\{\Delta(a)(1\!\!1 \otimes b) \mid a, b \in A\}.$$

$v_{kl}$ CAN BE ANY MONOMIAL IN THE GENERATORS OF $A$, SO FOR ANY $x$ IN THE (DENSE) $*$-ALGEBRA GENERATED BY $\alpha, \gamma$ AND ANY $c \in A$

$$x \otimes c \in \mathrm{span}\{\Delta(a)(1\!\!1 \otimes b) \mid a, b \in A\}. \qquad \blacksquare$$

<u>FACT:</u> $\mathrm{span}\{(a \otimes 1\!\!1)\Delta(b) \mid a, b \in A\}$ IS DENSE IN $A$.

(SIMILAR PROOF)

THEOREM (S.L. WORONOWICZ, A. VAN DAELE):

LET $A$ BE A UNITAL $C^*$-ALGEBRA, $\Delta : A \to A \otimes A$ A UNITAL $*$-HOMOMORPHISM SUCH THAT

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta .$$

ASSUME THAT

$$\mathrm{span}\{\Delta(a)(1\!\!1 \otimes b) \mid a, b \in A\} \quad \text{AND} \quad \mathrm{span}\{(a \otimes 1\!\!1)\Delta(b) \mid a, b \in A\}$$

ARE DENSE IN $A \otimes A$.

THEN THERE EXISTS A UNIQUE STATE $h$ ON $A$ SUCH THAT

$$\left.\begin{aligned} (id \otimes h)\, \Delta(a) &= h(a) 1\!\!1 \\ (h \otimes id)\, \Delta(a) &= h(a) 1\!\!1 \end{aligned}\right. \qquad \forall a \in A$$

- IN OUR CASE WE HAVE

$$h(a) = (1-q^2) \sum_{n=0}^{\infty} q^{2n} \left( e_{n,0} \mid \pi(a)\, e_{n,0} \right)$$

$$\begin{aligned} \pi(\alpha)\, e_{n,k} &= \sqrt{1-q^{2n}}\, e_{n-1,k} \\ \pi(\gamma)\, e_{n,k} &= q^n\, e_{n,k+1} \end{aligned} \qquad (n \in \mathbb{Z}_+, \ k \in \mathbb{Z})$$

EXERCISE: DEFINE
$$\alpha_k = \begin{cases} \alpha^k & k \geq 0 \\ (\alpha^*)^{-k} & k < 0 \end{cases}$$

(a) $\mathrm{span}\{\alpha_k \gamma^n (\gamma^*)^m \mid k \in \mathbb{Z}, \ m, n \in \mathbb{Z}_+\}$ IS DENSE IN $A$

(b) $h(\alpha_k \gamma^n (\gamma^*)^m) = \delta_{k,0}\, \delta_{m,n}\, \dfrac{1-q^2}{1-q^{2n+2}}$.

- THERE EXISTS A UNIQUE CHARACTER $e : A \to \mathbb{C}$ SUCH THAT $(id \otimes e) \circ \Delta = id = (e \otimes id) \circ \Delta$
- LET $\mathcal{A} = \operatorname{span}\{ \alpha_k \gamma^n (\gamma^*)^m \mid k \in \mathbb{Z},\ m, n \in \mathbb{Z}_+ \} \subset A$. THEN $\mathcal{A}$ IS A DENSE UNITAL $*$-SUBALGEBRA OF $A$ ($\mathcal{A}$ IS THE $*$-ALGEBRA GENERATED BY $\alpha$ AND $\gamma$)
  THERE EXISTS A UNIQUE LINEAR ANTI-MULTIPLICATIVE

$$\kappa : \mathcal{A} \longrightarrow \mathcal{A}$$

SUCH THAT

$$\kappa(\alpha) = \alpha^*, \qquad \kappa(\alpha^*) = \alpha,$$
$$\kappa(\gamma) = -q\gamma, \qquad \kappa(\gamma^*) = -q^{-1}\gamma^*.$$

WE HAVE

$$(id_{M_2} \otimes \kappa)\, u = u^* \quad (= u^{-1}).$$

- FOR $n \in \frac{1}{2}\mathbb{N}$ LET $T_n = \{-n, -n+1, \dots, n\}$.
  FOR $k \in T_n$ DEFINE $x_k = \alpha^{n+k} (\gamma^*)^{n-k}$.
  IT CAN BE SHOWN THAT

$$\Delta(x_k) = \sum_{i \in T_n} x_i \otimes w_{i,k}$$

FOR UNIQUE $w_{i,k} \in \mathcal{A}$ $(i, k \in T_n)$.
WE HAVE THE $(2n+1) \times (2n+1)$ MATRIX

$$w = \begin{pmatrix} w_{-n,-n} & \cdots & w_{-n,n} \\ \vdots & & \vdots \\ w_{n,-n} & \cdots & w_{n,n} \end{pmatrix}$$

AND

$$(id_{M_{2n+1}} \otimes \Delta)\, w = w_1 w_2 .$$