Quantum groups from analytic viewpoint

Piotr M. Sołtan

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❶ **From groups to quantum groups – motivation**

❷ **Examples**

❸ **Typical problems**

❹ **Multiplicative unitaries**

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Topology	Algebra

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Now let us forget that A was commutative!

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Quantum "$az + b$" group

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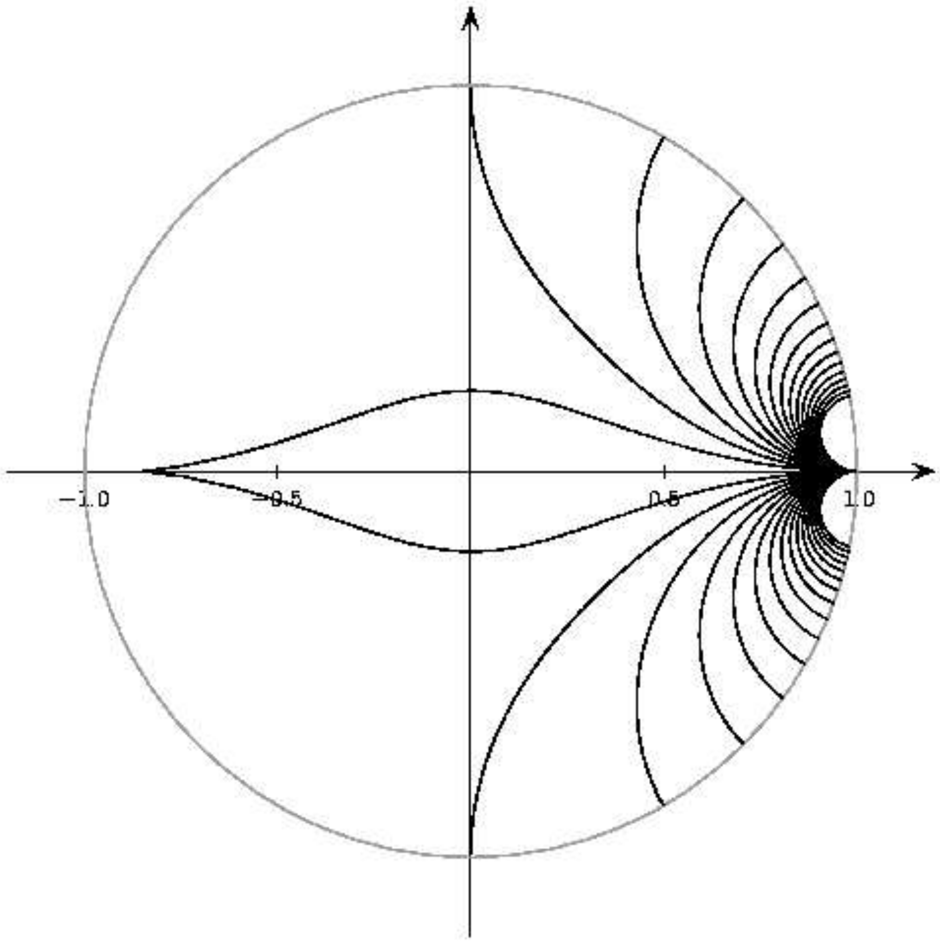

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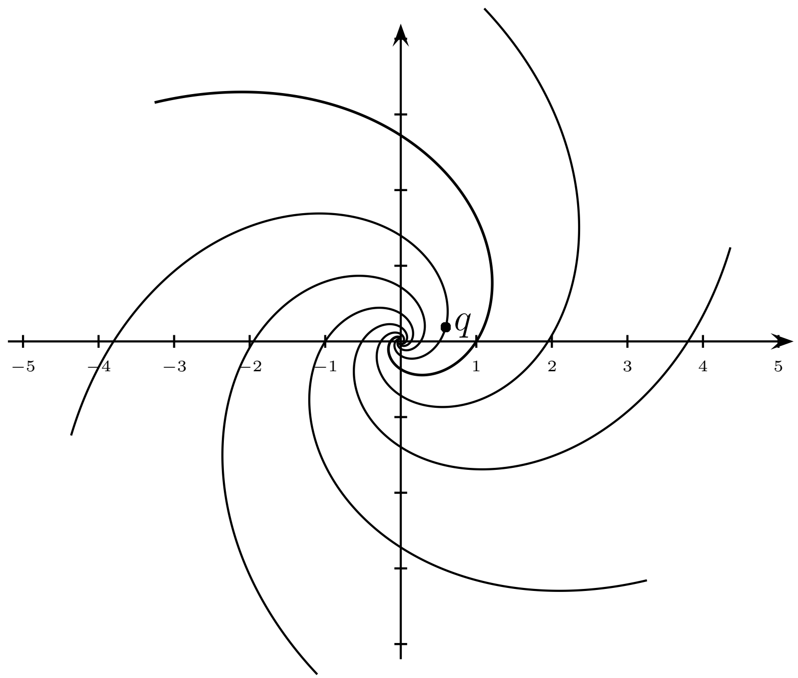

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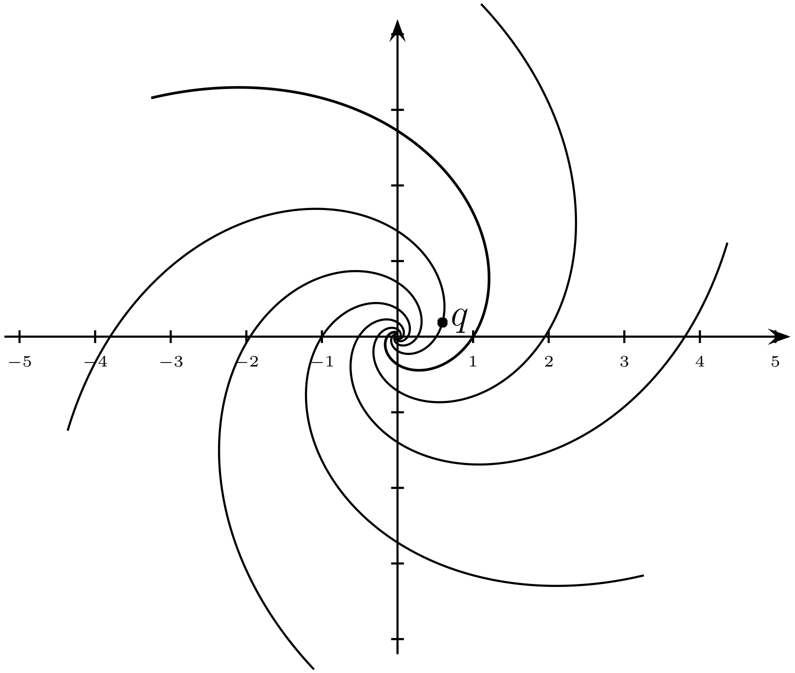

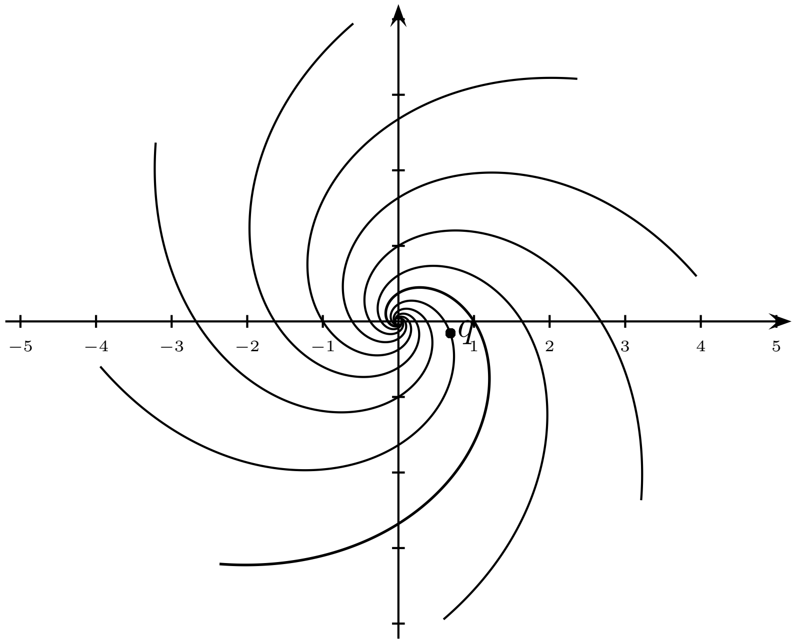

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➪ Need for advanced technical tools of functional analysis

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- ➪ the operator $a \otimes b + b \otimes I$ is not closed and thus not normal
- ➪ it is closable and its closure $a \otimes b \dot{+} b \otimes I$ is normal
- ➪ it would not have any normal extensions if spectra of a and b were different

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➪ Choose $0 < q < 1$, consider α, γ such that

$$\begin{bmatrix} \alpha & q\gamma^* \\ \gamma & \alpha^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \alpha^* & \gamma^* \\ q\gamma & \alpha \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

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➪ Choose $0 < q < 1$, consider α, γ such that

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The problem has since been (partially) solved by Korogodsky, Woronowicz, Kustermans, Koelink.

Multiplicative unitaries

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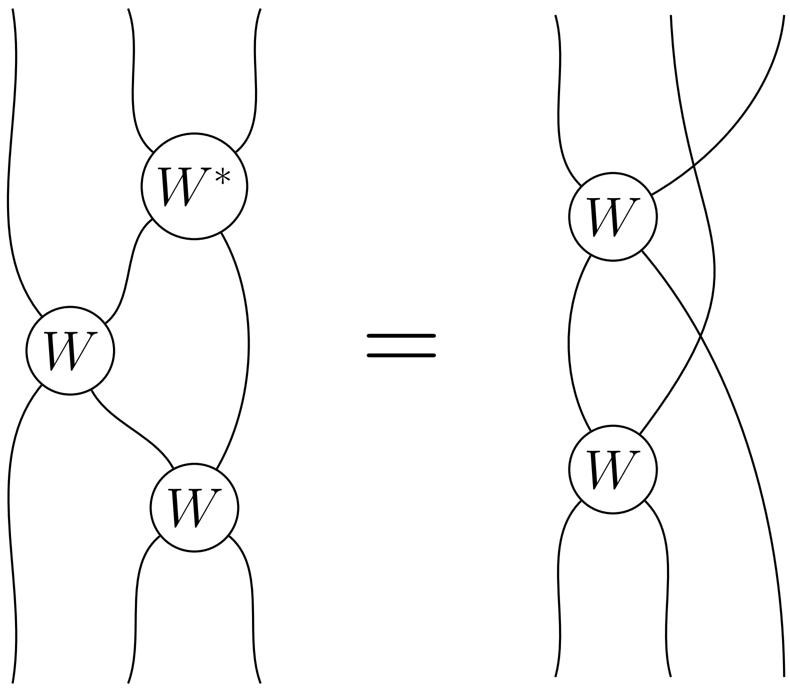

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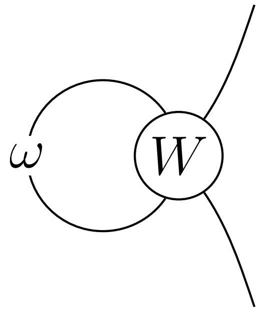

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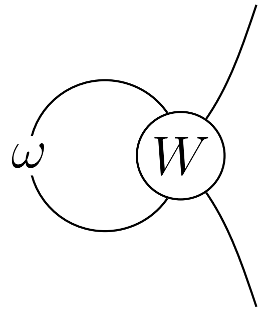
$A = \{$ 

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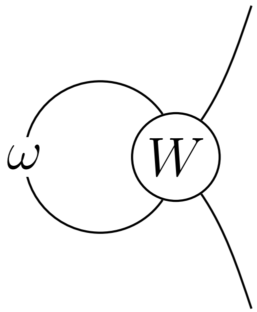
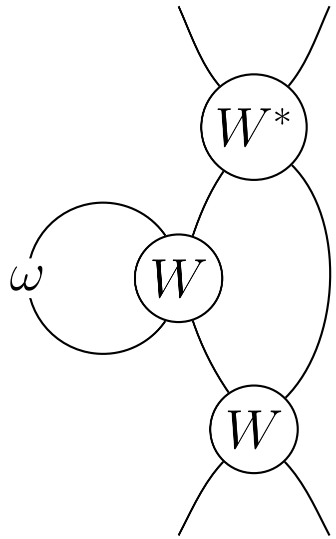
$\Delta\big($ 

$\big) =$ 

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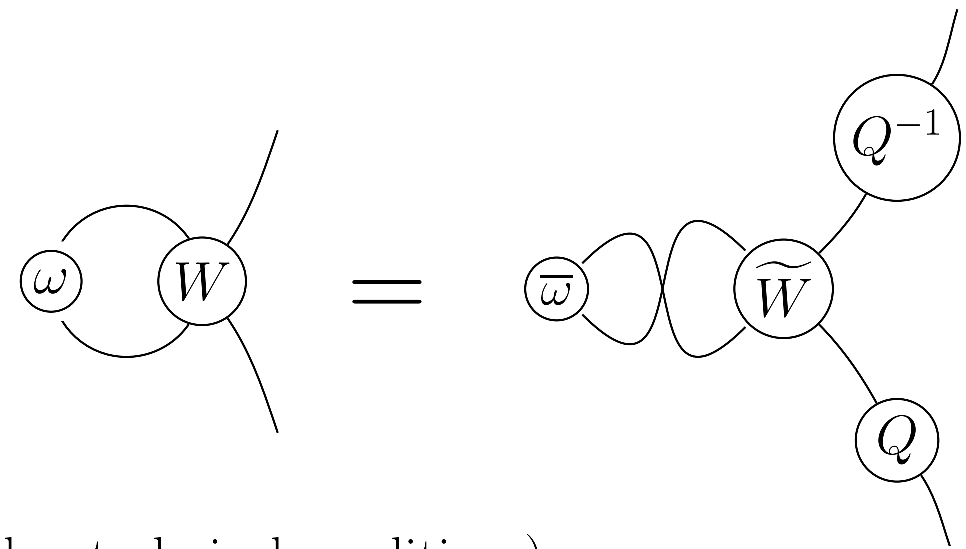
$\exists$ unitary $\widetilde{W}$ and positive $Q = Q^*$ such that



(plus some other technical conditions)

4

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- The map

$$\Delta : A \ni a \longmapsto W(a \otimes I)W^* \in \mathrm{M}(A \otimes A)$$

is the standard comultiplication on $\mathrm{C}_0(\mathbb{G})$.

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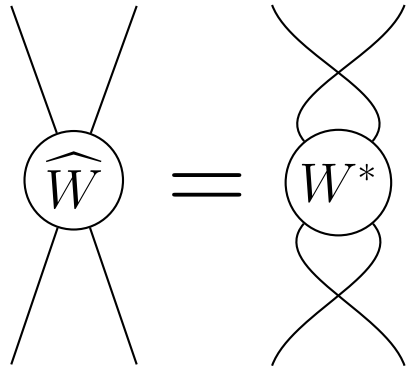

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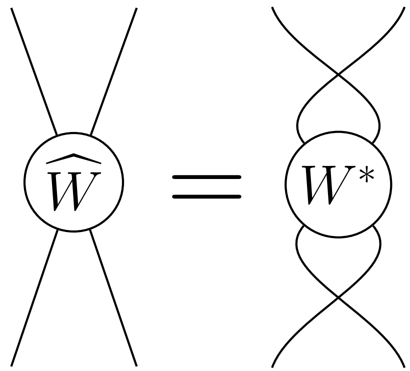

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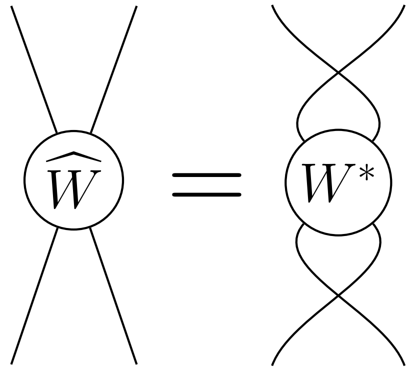

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- ➪ the dual of quantum "$az + b$" is its opposite quantum group

❹

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- ➪ Representation theory can be studied in this language (Woronowicz, P.M.S.)
- ➪ Modularity gives a new framework to study existence of Haar measures (Haar weights)

Thank you