Quantum Bohr compactification

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Classical Bohr compactification

G — **locally compact group**

Thm.: $\exists!$ **compact group** $\overline{G}$ **and a continuous homomorphism**

$$\alpha\colon G \longrightarrow \overline{G}$$

such that for any continuous homomorphism

$$\beta\colon G \longrightarrow K$$

from G **to a compact group** K **there is a unique**

$$\overline{\beta}\colon \overline{G} \longrightarrow K$$

such that

$$\begin{array}{ccc} G & \xrightarrow{\beta} & K \\ {\scriptstyle\alpha}\downarrow & \nearrow{\scriptstyle\overline{\beta}} & \\ \overline{G} & & \end{array}$$

$$\left\{ \begin{array}{c}\textbf{almost periodic} \\ \textbf{functions on } G\end{array} \right\} = \left\{ \textbf{ functions on } \overline{G} \right\}$$

Algebraic Discrete Quantum Groups

$$\mathscr{A} = \bigoplus_{\alpha \in \mathscr{R}} M_{N^\alpha} \qquad \textbf{(algebraic)}$$

$$\delta \colon \mathscr{A} \to \mathscr{M}(\mathscr{A} \otimes_{\mathrm{alg}} \mathscr{A}) \qquad \textbf{(non degenerate)}$$

The maps

$$T_1 : a \otimes b \longmapsto \delta(a)(I \otimes b)$$
$$T_2 : a \otimes b \longmapsto (a \otimes I)\delta(b)$$

are bijections $\mathscr{A} \otimes_{\mathrm{alg}} \mathscr{A} \longrightarrow \mathscr{A} \otimes_{\mathrm{alg}} \mathscr{A}$

δ **is coassociative, i.e.**

$$(T_2 \otimes \mathrm{id})(a \otimes b \otimes c) = (\mathrm{id} \otimes T_1)(a \otimes b \otimes c)$$

Thm.: $\exists\ h_{\mathrm{R}},\ h_{\mathrm{L}},\ e \colon \mathscr{A} \to \mathbb{C}$, $\kappa \colon \mathscr{A} \to \mathscr{A}$

Dual quantum group is compact

C^*-Algebraic Discrete Quantum Groups

$$A = \bigoplus_{\alpha \in \mathscr{R}} M_{N^\alpha} \qquad (\mathrm{C}^*\textbf{-algebraic})$$

$$\delta \in \mathrm{Mor}(A, A \otimes A)$$

The sets

$$\Big\{\delta(a)(I \otimes b) : \ a, b \in A\Big\}$$
$$\Big\{(a \otimes I)\delta(b) : \ a, b \in A\Big\}$$

are linearly dense in $A \otimes A$

δ **is coassociative, i.e.**

$$(\delta \otimes \mathrm{id})\circ\delta = (\mathrm{id} \otimes \delta)\circ\delta$$

Thm.: $\exists\ h_{\mathrm{R}},\ h_{\mathrm{L}},\ e\colon \mathscr{A} \to \mathbb{C},\ \kappa\colon \mathscr{A} \to \mathscr{A}$

Dual quantum group is compact

Quantum Bohr compactification (algebraic)

$(\mathscr{A},\delta)$ — algebraic discrete quantum group

Def.: Almost Periodic Elements

$x \in \mathscr{M}(\mathscr{A})$ **is *almost periodic* if**

$$\delta(x) \in \mathscr{M}(\mathscr{A}) \otimes_{\mathrm{alg}} \mathscr{M}(\mathscr{A})$$

The set of almost periodic elements is denoted by $\mathscr{AP}$

Thm.:

1. $\mathscr{AP}$ **is a unital $*$-subalgebra of** $\mathscr{M}(\mathscr{A})$
2. $\delta(\mathscr{AP}) \subset \mathscr{AP} \otimes_{\mathrm{alg}} \mathscr{AP}$
3. $\kappa(\mathscr{AP}) = \mathscr{AP}$
4. with δ, e and κ from $\mathscr{M}(\mathscr{A})$, $\mathscr{AP}$ is a Hopf $*$-algebra

- **expected universal property** $\smile$
- **no mean for almost periodic elements** $\frown$

Problem

Lack of mean, no hope for a compact quantum group in passage to C^*-algebraic level.

Solution

Use bounded almost periodic elements:

(A, δ) — C^***-algebraic discrete quantum group**

$(\mathscr{A}, \delta)$ — **Pedersen ideal (algebraic discrete quantum group)**

$(\mathscr{AP}, \delta)$ — **(algebraic) Bohr compactification of** $(\mathscr{A}, \delta)$

$$\mathbb{AP} = \textbf{closure of } \mathrm{M}(A) \cap \mathscr{AP}$$

makes sense, since

$$\mathscr{AP} \subset \mathscr{M}(\mathscr{A}) = A^\eta$$
$$\mathrm{M}(A) \subset A^\eta$$

Unimodular Case

Thm.: (A, δ) — **unimodular**

1. $\mathbb{AP}$ **is a unital** $*$**-subalgebra of** $\mathrm{M}(A)$
2. $\delta(\mathbb{AP}) \subset \mathbb{AP} \otimes \mathbb{AP}$
3. $(\mathbb{AP}, \delta)$ **is a compact quantum group**
4. $\mathbb{AP}$ **is the closed linear span of matrix elements of finite dimensional unitary representations of** (A, δ)

Genearal Case

Thm.: **We** ***define*** $\mathbb{AP}$ **as the closed linear span of matrix elements of finite dimensional unitary representations of** (A, δ)

1. $\mathbb{AP}$ **is a unital** $*$**-subalgebra of** $\mathrm{M}(A)$
2. $\delta(\mathbb{AP}) \subset \mathbb{AP} \otimes \mathbb{AP}$
3. $(\mathbb{AP}, \delta)$ **is a compact quantum group**

Features

- proof uses new tool — the Canonical Kac Quotient
- the expected universal property
- $\exists$ a number of simple examples
- construction leads to "profinite quantum groups"
- notion of Maximal Almost Periodicity investigated