Algebraic origin of the Jones polynomial

Piotr M. Sołtan

Institute of Mathematics of the Polish Academy of Sciences
and
Department of Mathematical Methods in Physics, Faculty of Physics,
Warsaw University

October 22, 2010

Knots and links

Knots and links

- A **link** is a finite family of circles embedded in $\mathbb{R}^3$ (or $\mathbb{S}^3$).

Knots and links

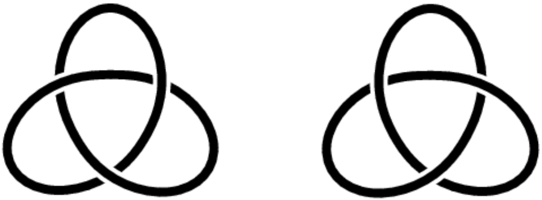
- A **link** is a finite family of circles embedded in $\mathbb{R}^3$ (or $\mathbb{S}^3$).
- A **knot** is a link with only one component.

Knots and links

- A **link** is a finite family of circles embedded in $\mathbb{R}^3$ (or $\mathbb{S}^3$).
- A **knot** is a link with only one component.

KNOTS AND LINKS

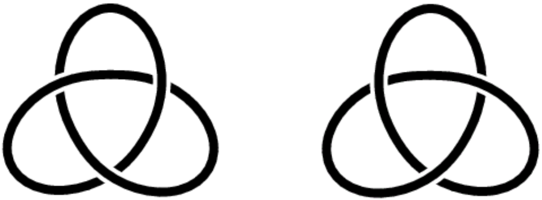
- A **link** is a finite family of circles embedded in $\mathbb{R}^3$ (or $\mathbb{S}^3$).
- A **knot** is a link with only one component.



- These are **knot (link) diagrams**.

Knots and links

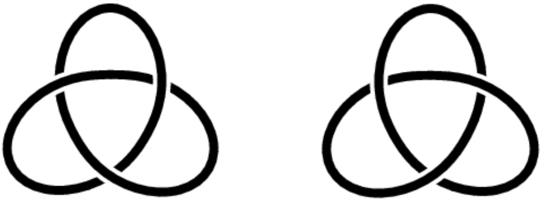
- A **link** is a finite family of circles embedded in $\mathbb{R}^3$ (or $\mathbb{S}^3$).
- A **knot** is a link with only one component.



- These are **knot (link) diagrams**.
- We are working in the PL-category.

Knots and links

- A **link** is a finite family of circles embedded in $\mathbb{R}^3$ (or $\mathbb{S}^3$).
- A **knot** is a link with only one component.



- These are **knot (link) diagrams**.
- We are working in the PL-category.
- It is preferable to work with **oriented links** (each circle is oriented).

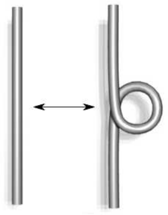
Equivalence of links

Equivalence of links

- Two links are **equivalent** if their diagrams are related via **Reidemeister moves**

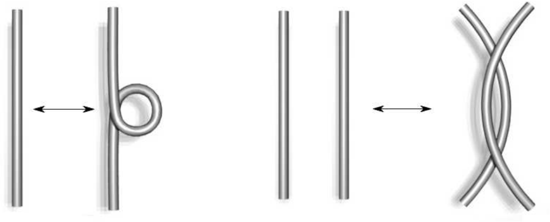
Equivalence of links

- Two links are **equivalent** if their diagrams are related via **Reidemeister moves**:



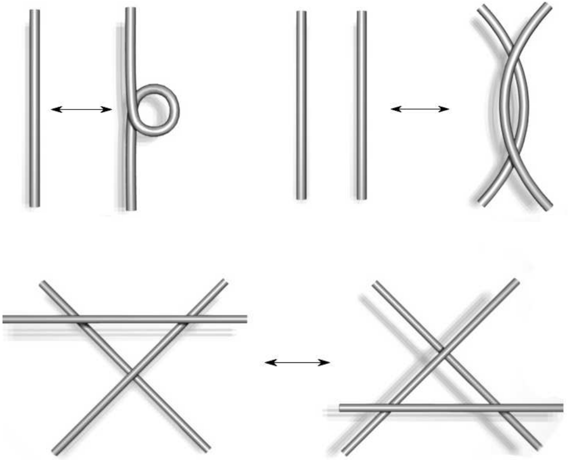
Equivalence of links

- Two links are **equivalent** if their diagrams are related via **Reidemeister moves**:



Equivalence of links

- Two links are **equivalent** if their diagrams are related via **Reidemeister moves**:



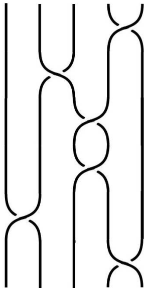
Braids

Braids

- Braids can be visualized as diagrams

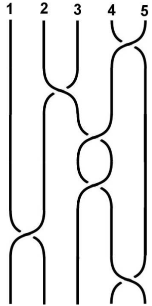
Braids

- Braids can be visualized as diagrams:



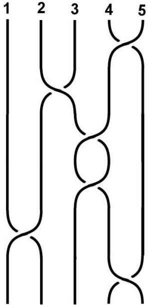
Braids

- Braids can be visualized as diagrams:



Braids

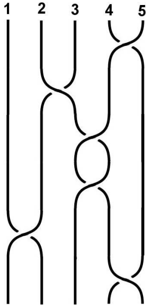
- Braids can be visualized as diagrams:



- The set of braids on n strands forms a group B_n

Braids

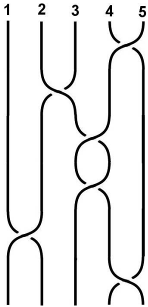
- Braids can be visualized as diagrams:



- The set of braids on n strands forms a group B_n:
  - the group operation is juxtaposition of diagrams,

Braids

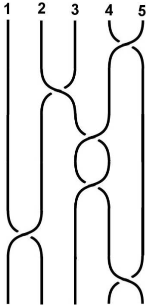
- Braids can be visualized as diagrams:



- The set of braids on n strands forms a group B_n:
  - the group operation is juxtaposition of diagrams,
  - the unit is the diagram with no crossings,

Braids

- Braids can be visualized as diagrams:



- The set of braids on n strands forms a group B_n:
  - the group operation is juxtaposition of diagrams,
  - the unit is the diagram with no crossings,
  - the inverse is the reflection in a horizontal line.

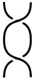
Braids

Braids

- We identify braids whose diagrams are identical except for a part containing one of these fragments:

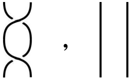
Braids

- We identify braids whose diagrams are identical except for a part containing one of these fragments:



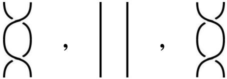
Braids

- We identify braids whose diagrams are identical except for a part containing one of these fragments:



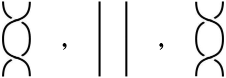
Braids

- We identify braids whose diagrams are identical except for a part containing one of these fragments:

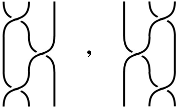


Braids

- We identify braids whose diagrams are identical except for a part containing one of these fragments:

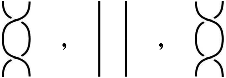


and

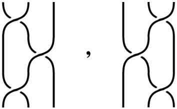


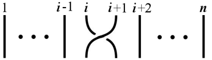
BRAIDS

- We identify braids whose diagrams are identical except for a part containing one of these fragments:



and

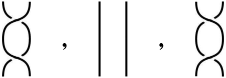


- Let $\sigma_i =$ 

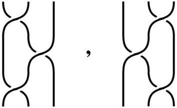
.

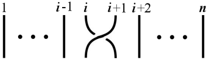
Braids

- We identify braids whose diagrams are identical except for a part containing one of these fragments:



and

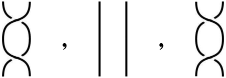


- Let $\sigma_i =$ 

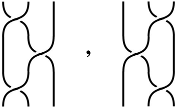
.
- B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$

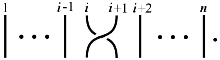
Braids

- We identify braids whose diagrams are identical except for a part containing one of these fragments:



and

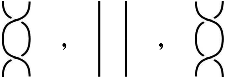


- Let $\sigma_i =$ 

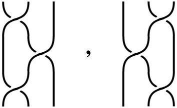
- B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$ with only these relations

Braids

- We identify braids whose diagrams are identical except for a part containing one of these fragments:



and

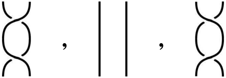


- Let $\sigma_i =$ 

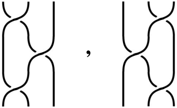
- B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$ with only these relations:
  - $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$,

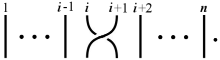
Braids

- We identify braids whose diagrams are identical except for a part containing one of these fragments:



and



- Let $\sigma_i =$ 

- B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$ with only these relations:
  - $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$,
  - $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $i = 1, \ldots, n-2$.

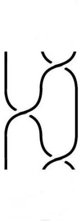
From braids to links and back

From braids to links and back

- From braids to links

From braids to links and back

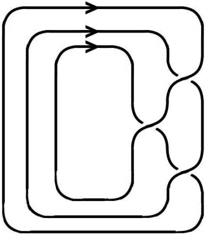
- From braids to links:



$$= \alpha$$

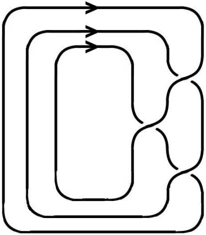
From braids to links and back

- From braids to links:

 $= \widehat{\alpha}$

From braids to links and back

- From braids to links:

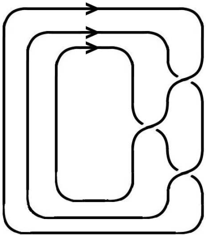


$= \widehat{\alpha}$

(the operation of **closure**).

From braids to links and back

- From braids to links:



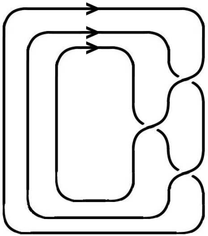
$$= \widehat{\alpha}$$

(the operation of **closure**).

- From links to braids

From braids to links and back

- From braids to links:

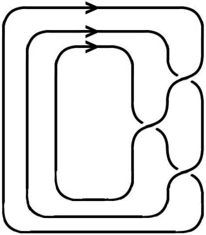
 $= \widehat{\alpha}$

  (the operation of **closure**).

- From links to braids: any link can be represented as a closed braid

From braids to links and back

- From braids to links:



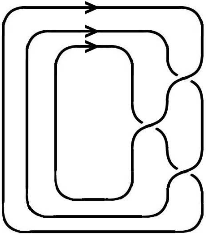
$= \widehat{\alpha}$

(the operation of **closure**).

- From links to braids: any link can be represented as a closed braid

From braids to links and back

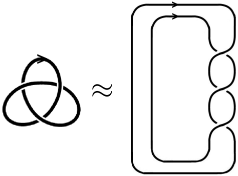
- From braids to links:



$$= \widehat{\alpha}$$

  (the operation of **closure**).

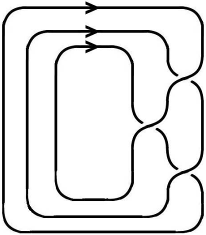
- From links to braids: any link can be represented as a closed braid



$$= \widehat{\sigma_1^3}$$

From braids to links and back

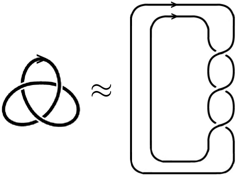
- From braids to links:



$= \widehat{\alpha}$

(the operation of **closure**).

- From links to braids: any link can be represented as a closed braid



$\approx$ … $= \widehat{\sigma_1^3}$

(this is **Alexander's theorem**).

When closures of braids are equivalent?

When closures of braids are equivalent?

- Let $\alpha \in B_n$, $\beta \in B_m$.

When closures of braids are equivalent?

- Let $\alpha \in B_n$, $\beta \in B_m$.
- $\widehat{\alpha} \approx \widehat{\beta}$

When closures of braids are equivalent?

- Let $\alpha \in B_n$, $\beta \in B_m$.
- $\widehat{\alpha} \approx \widehat{\beta}$ if and only if

When closures of braids are equivalent?

- Let $\alpha \in B_n$, $\beta \in B_m$.
- $\widehat{\alpha} \approx \widehat{\beta}$ if and only if β can be obtained from α through **Markov moves**

When closures of braids are equivalent?

- Let $\alpha \in B_n$, $\beta \in B_m$.
- $\widehat{\alpha} \approx \widehat{\beta}$ if and only if β can be obtained from α through **Markov moves**:
  - conjugation

$$B_k \ni \gamma \longmapsto \delta\gamma\delta^{-1} \in B_k$$

When closures of braids are equivalent?

- Let $\alpha \in B_n$, $\beta \in B_m$.
- $\widehat{\alpha} \approx \widehat{\beta}$ if and only if β can be obtained from α through **Markov moves**:
  - conjugation

$$B_k \ni \gamma \longmapsto \delta\gamma\delta^{-1} \in B_k$$

    (for some $\delta \in B_k$),

WHEN CLOSURES OF BRAIDS ARE EQUIVALENT?

- Let $\alpha \in B_n$, $\beta \in B_m$.
- $\widehat{\alpha} \approx \widehat{\beta}$ if and only if β can be obtained from α through **Markov moves**:
  - conjugation

    $$B_k \ni \gamma \longmapsto \delta\gamma\delta^{-1} \in B_k$$

    (for some $\delta \in B_k$),
  - passage to different group via

    $$B_k \ni \delta \longmapsto \delta\sigma_k^{\pm 1} \in B_{k+1}$$

When closures of braids are equivalent?

- Let $\alpha \in B_n$, $\beta \in B_m$.
- $\widehat{\alpha} \approx \widehat{\beta}$ if and only if β can be obtained from α through **Markov moves**:
  - conjugation
    $$B_k \ni \gamma \longmapsto \delta\gamma\delta^{-1} \in B_k$$
    (for some $\delta \in B_k$),
  - passage to different group via
    $$B_k \ni \delta \longmapsto \delta\sigma_k^{\pm 1} \in B_{k+1}$$

and inverses of these operations

When closures of braids are equivalent?

- Let $\alpha \in B_n$, $\beta \in B_m$.
- $\widehat{\alpha} \approx \widehat{\beta}$ if and only if β can be obtained from α through **Markov moves**:
  - conjugation
    $$B_k \ni \gamma \longmapsto \delta\gamma\delta^{-1} \in B_k$$
    (for some $\delta \in B_k$),
  - passage to different group via
    $$B_k \ni \delta \longmapsto \delta\sigma_k^{\pm 1} \in B_{k+1}$$

  and inverses of these operations (this is **Markov's theorem**).

When closures of braids are equivalent?

- Let $\alpha \in B_n$, $\beta \in B_m$.
- $\widehat{\alpha} \approx \widehat{\beta}$ if and only if β can be obtained from α through **Markov moves**:
  - conjugation
    $$B_k \ni \gamma \longmapsto \delta\gamma\delta^{-1} \in B_k$$
    (for some $\delta \in B_k$),
  - passage to different group via
    $$B_k \ni \delta \longmapsto \delta\sigma_k^{\pm 1} \in B_{k+1}$$

  and inverses of these operations (this is **Markov's theorem**).
- This shows that we need to consider

  $B_\infty = \varinjlim B_n$

When closures of braids are equivalent?

- Let $\alpha \in B_n$, $\beta \in B_m$.
- $\widehat{\alpha} \approx \widehat{\beta}$ if and only if β can be obtained from α through **Markov moves**:
  - conjugation
  $$B_k \ni \gamma \longmapsto \delta\gamma\delta^{-1} \in B_k$$
  (for some $\delta \in B_k$),
  - passage to different group via
  $$B_k \ni \delta \longmapsto \delta\sigma_k^{\pm 1} \in B_{k+1}$$

  and inverses of these operations (this is **Markov's theorem**).
- This shows that we need to consider $B_\infty = \varinjlim B_n = \bigcup_{n=1}^{\infty} B_n$.

Illustration of Markov moves

Illustration of Markov moves

- Markov move of type I

Illustration of Markov moves

- Markov move of type I:

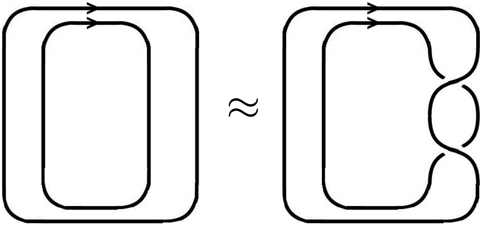
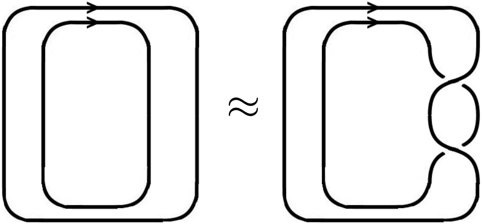


Illustration of Markov moves

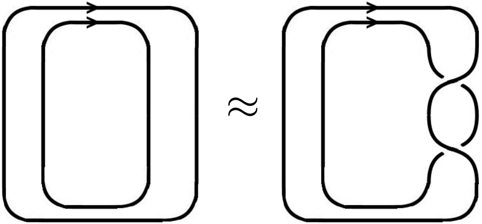
- Markov move of type I:



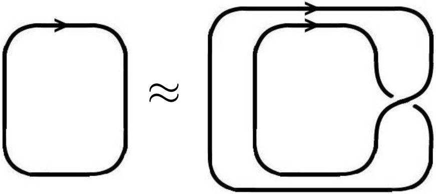
- Markov move of type II

Illustration of Markov moves

- Markov move of type I:



- Markov move of type II:



How to get link invariants

How to get link invariants

Strategy:

How to get link invariants

Strategy:

- Take a link L,

How to get link invariants

Strategy:

- Take a link L,
- represent L as a closed braid $\widehat{\alpha}$ with $\alpha \in B_n \subset B_\infty$,

How to get link invariants

Strategy:

- Take a link L,
- represent L as a closed braid $\widehat{\alpha}$ with $\alpha \in B_n \subset B_\infty$,
- represent B_∞ in some algebra $\mathcal{H}$,

How to get link invariants

Strategy:

- Take a link L,
- represent L as a closed braid $\widehat{\alpha}$ with $\alpha \in B_n \subset B_\infty$,
- represent B_∞ in some algebra $\mathcal{H}$,
- find a function φ on $\mathcal{H}$ which is invariant for Markov's moves (depending on some parameters),

How to get link invariants

Strategy:

- Take a link L,
- represent L as a closed braid $\widehat{\alpha}$ with $\alpha \in B_n \subset B_\infty$,
- represent B_∞ in some algebra $\mathcal{H}$,
- find a function φ on $\mathcal{H}$ which is invariant for Markov's moves (depending on some parameters),
- calculate the value of φ on the image of α in $\mathcal{H}$ (e.g. as polynomial expression in the parameters).

How to get link invariants

STRATEGY:

- Take a link L,
- represent L as a closed braid $\widehat{\alpha}$ with $\alpha \in B_n \subset B_\infty$,
- represent B_∞ in some algebra $\mathcal{H}$,
- find a function φ on $\mathcal{H}$ which is invariant for Markov's moves (depending on some parameters),
- calculate the value of φ on the image of α in $\mathcal{H}$ (e.g. as polynomial expression in the parameters).

QUESTIONS:

How to get link invariants

Strategy:

- Take a link L,
- represent L as a closed braid $\widehat{\alpha}$ with $\alpha \in B_n \subset B_\infty$,
- represent B_∞ in some algebra $\mathcal{H}$,
- find a function φ on $\mathcal{H}$ which is invariant for Markov's moves (depending on some parameters),
- calculate the value of φ on the image of α in $\mathcal{H}$ (e.g. as polynomial expression in the parameters).

Questions:

- What do we take for $\mathcal{H}$?

How to get link invariants

Strategy:

- Take a link L,
- represent L as a closed braid $\widehat{\alpha}$ with $\alpha \in B_n \subset B_\infty$,
- represent B_∞ in some algebra $\mathcal{H}$,
- find a function φ on $\mathcal{H}$ which is invariant for Markov's moves (depending on some parameters),
- calculate the value of φ on the image of α in $\mathcal{H}$ (e.g. as polynomial expression in the parameters).

Questions:

- What do we take for $\mathcal{H}$?
- What do we take for φ?

How to get link invariants

Strategy:

- Take a link L,
- represent L as a closed braid $\widehat{\alpha}$ with $\alpha \in B_n \subset B_\infty$,
- represent B_∞ in some algebra $\mathcal{H}$,
- find a function φ on $\mathcal{H}$ which is invariant for Markov's moves (depending on some parameters),
- calculate the value of φ on the image of α in $\mathcal{H}$ (e.g. as polynomial expression in the parameters).

Questions:

- What do we take for $\mathcal{H}$?
- What do we take for φ?
- Is our result nontrivial (examples)?

How to get link invariants

Strategy:

- Take a link L,
- represent L as a closed braid $\widehat{\alpha}$ with $\alpha \in B_n \subset B_\infty$,
- represent B_∞ in some algebra $\mathcal{H}$,
- find a function φ on $\mathcal{H}$ which is invariant for Markov's moves (depending on some parameters),
- calculate the value of φ on the image of α in $\mathcal{H}$ (e.g. as polynomial expression in the parameters).

Questions:

- What do we take for $\mathcal{H}$?
- What do we take for φ?
- Is our result nontrivial (examples)?
- Is the invariant easily computable in some other way?

The Hecke algebra

The Hecke algebra

- Let q be a complex parameter.

The Hecke algebra

- Let q be a complex parameter.
- Let $\mathcal{H}_n$ be the algebra generated by elements $g_1, \dots, g_{n-1}$

The Hecke algebra

- Let q be a complex parameter.
- Let $\mathcal{H}_n$ be the algebra generated by elements $g_1, \ldots, g_{n-1}$ with relations
  1. $g_1^2 = (1-q)g_i + q \qquad \text{for } i = 1, \ldots, n-1,$

The Hecke algebra

- Let q be a complex parameter.
- Let $\mathcal{H}_n$ be the algebra generated by elements $g_1, \ldots, g_{n-1}$ with relations
    1. $g_1^2 = (1-q)g_i + q \quad$ for $i = 1, \ldots, n-1$,
    2. $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad$ for $i = 1, \ldots, n-2$,

The Hecke algebra

- Let q be a complex parameter.
- Let $\mathcal{H}_n$ be the algebra generated by elements $g_1, \ldots, g_{n-1}$ with relations
  1. $g_1^2 = (1-q)g_i + q$ for $i = 1, \ldots, n-1$,
  2. $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $i = 1, \ldots, n-2$,
  3. $g_i g_j = g_j g_i$ for $|i-j| \geq 2$.

The Hecke algebra

- Let q be a complex parameter.
- Let $\mathcal{H}_n$ be the algebra generated by elements $g_1, \ldots, g_{n-1}$ with relations
  1. $g_1^2 = (1-q)g_i + q$ for $i = 1, \ldots, n-1$,
  2. $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $i = 1, \ldots, n-2$,
  3. $g_i g_j = g_j g_i$ for $|i-j| \geq 2$.
- Let $\mathcal{H} = \varinjlim \mathcal{H}_n$

The Hecke algebra

- Let q be a complex parameter.
- Let $\mathcal{H}_n$ be the algebra generated by elements $g_1, \ldots, g_{n-1}$ with relations
  1. $g_1^2 = (1-q)g_i + q$ for $i = 1, \ldots, n-1$,
  2. $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $i = 1, \ldots, n-2$,
  3. $g_i g_j = g_j g_i$ for $|i-j| \geq 2$.
- Let $\mathcal{H} = \varinjlim \mathcal{H}_n = \bigcup_{n=1}^{\infty} \mathcal{H}_n$.

The Hecke algebra

- Let q be a complex parameter.
- Let $\mathcal{H}_n$ be the algebra generated by elements $g_1, \ldots, g_{n-1}$ with relations
  1. $g_1^2 = (1-q)g_i + q$ for $i = 1, \ldots, n-1$,
  2. $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $i = 1, \ldots, n-2$,
  3. $g_i g_j = g_j g_i$ for $|i-j| \geq 2$.
- Let $\mathcal{H} = \varinjlim \mathcal{H}_n = \bigcup_{n=1}^{\infty} \mathcal{H}_n$.
- Clearly we have representation

$$B_n \ni \sigma_i \longmapsto g_i \in \mathcal{H}_n$$

The Hecke algebra

- Let q be a complex parameter.
- Let $\mathcal{H}_n$ be the algebra generated by elements $g_1, \ldots, g_{n-1}$ with relations
  1. $g_1^2 = (1-q)g_i + q$ for $i = 1, \ldots, n-1$,
  2. $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $i = 1, \ldots, n-2$,
  3. $g_i g_j = g_j g_i$ for $|i-j| \geq 2$.
- Let $\mathcal{H} = \varinjlim \mathcal{H}_n = \bigcup_{n=1}^{\infty} \mathcal{H}_n$.
- Clearly we have representation

$$B_n \ni \sigma_i \longmapsto g_i \in \mathcal{H}_n$$

which extends to a representation

$$\pi : B_\infty \longrightarrow \mathcal{H}.$$

Ocneanu's functional

Ocneanu's functional

Theorem

For any $z \in \mathbb{C}$ *there exists a unique linear functional* $\mathrm{tr} : \mathcal{H} \to \mathbb{C}$ *such that*

Ocneanu's functional

Theorem

For any $z \in \mathbb{C}$ *there exists a unique linear functional* $\mathrm{tr} : \mathcal{H} \to \mathbb{C}$ *such that*

1. $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ *for any* $a, b \in \mathcal{H}$*,*

Ocneanu's functional

Theorem

For any $z \in \mathbb{C}$ *there exists a unique linear functional* $\mathrm{tr} : \mathcal{H} \to \mathbb{C}$ *such that*

1. $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ *for any* $a, b \in \mathcal{H}$,
2. $\mathrm{tr}(1) = 1$,

Ocneanu's functional

Theorem

For any $z \in \mathbb{C}$ *there exists a unique linear functional* $\mathrm{tr} : \mathcal{H} \to \mathbb{C}$ *such that*

1. $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ *for any* $a, b \in \mathcal{H}$,
2. $\mathrm{tr}(1) = 1$,
3. *for* $x \in \mathcal{H}_n \subset \mathcal{H}$ *we have*

$$\mathrm{tr}(x g_n) = z\,\mathrm{tr}(x).$$

Ocneanu's functional

Theorem

For any $z \in \mathbb{C}$ there exists a unique linear functional $\mathrm{tr} : \mathcal{H} \to \mathbb{C}$ such that

1. $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ *for any* $a, b \in \mathcal{H}$,
2. $\mathrm{tr}(1) = 1$,
3. *for* $x \in \mathcal{H}_n \subset \mathcal{H}$ *we have*

$$\mathrm{tr}(x g_n) = z\,\mathrm{tr}(x).$$

- Relations 1.–3. suffice to compute tr of any word on the generators (g_i).

Ocneanu's functional

Theorem

For any $z \in \mathbb{C}$ *there exists a unique linear functional* $\mathrm{tr} : \mathcal{H} \to \mathbb{C}$ *such that*

1. $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ *for any* $a, b \in \mathcal{H}$,
2. $\mathrm{tr}(1) = 1$,
3. *for* $x \in \mathcal{H}_n \subset \mathcal{H}$ *we have*

$$\mathrm{tr}(xg_n) = z\,\mathrm{tr}(x).$$

- Relations 1.–3. suffice to compute tr of any word on the generators (g_i). For example:

$\mathrm{tr}(g_2g_1g_3g_2) =$

Ocneanu's functional

Theorem

For any $z \in \mathbb{C}$ *there exists a unique linear functional* $\mathrm{tr} : \mathcal{H} \to \mathbb{C}$ *such that*

1. $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ *for any* $a, b \in \mathcal{H}$,
2. $\mathrm{tr}(1) = 1$,
3. *for* $x \in \mathcal{H}_n \subset \mathcal{H}$ *we have*

$$\mathrm{tr}(xg_n) = z\,\mathrm{tr}(x).$$

- Relations 1.–3. suffice to compute tr of any word on the generators (g_i). For example:

$$\mathrm{tr}(g_2g_1g_3g_2) = \mathrm{tr}(g_2^2g_1g_3) =$$

Ocneanu's functional

Theorem

For any $z \in \mathbb{C}$ there exists a unique linear functional $\mathrm{tr} : \mathcal{H} \to \mathbb{C}$ such that

1. $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ *for any* $a, b \in \mathcal{H}$,
2. $\mathrm{tr}(1) = 1$,
3. *for* $x \in \mathcal{H}_n \subset \mathcal{H}$ *we have*

$$\mathrm{tr}(xg_n) = z\,\mathrm{tr}(x).$$

- Relations 1.–3. suffice to compute tr of any word on the generators (g_i). For example:

$$\mathrm{tr}(g_2g_1g_3g_2) = \mathrm{tr}(g_2^2g_1g_3) = z\,\mathrm{tr}(g_2^2g_1) =$$

Ocneanu's functional

Theorem

For any $z \in \mathbb{C}$ there exists a unique linear functional $\mathrm{tr} : \mathcal{H} \to \mathbb{C}$ such that

1. $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ *for any* $a, b \in \mathcal{H}$,
2. $\mathrm{tr}(1) = 1$,
3. *for* $x \in \mathcal{H}_n \subset \mathcal{H}$ *we have*

$$\mathrm{tr}(x g_n) = z\,\mathrm{tr}(x).$$

- Relations 1.–3. suffice to compute tr of any word on the generators (g_i). For example:

$$\mathrm{tr}(g_2 g_1 g_3 g_2) = \mathrm{tr}(g_2^2 g_1 g_3) = z\,\mathrm{tr}(g_2^2 g_1) = z\,\mathrm{tr}\big(((q-1)g_2 + q)g_1\big)$$

$$=$$

Ocneanu's functional

Theorem

For any $z \in \mathbb{C}$ there exists a unique linear functional $\operatorname{tr} : \mathcal{H} \to \mathbb{C}$ such that

1. $\operatorname{tr}(ab) = \operatorname{tr}(ba)$ *for any* $a, b \in \mathcal{H}$,
2. $\operatorname{tr}(1) = 1$,
3. *for* $x \in \mathcal{H}_n \subset \mathcal{H}$ *we have*

$$\operatorname{tr}(x g_n) = z \operatorname{tr}(x).$$

- Relations 1.–3. suffice to compute tr of any word on the generators (g_i). For example:

$$\begin{aligned}\operatorname{tr}(g_2 g_1 g_3 g_2) &= \operatorname{tr}(g_2^2 g_1 g_3) = z \operatorname{tr}(g_2^2 g_1) = z \operatorname{tr}\big(((q-1)g_2 + q)g_1\big) \\ &= z(q-1)\operatorname{tr}(g_2 g_1) + zq \operatorname{tr}(g_1) \\ &= \end{aligned}$$

Ocneanu's functional

Theorem

For any $z \in \mathbb{C}$ there exists a unique linear functional $\mathrm{tr} : \mathcal{H} \to \mathbb{C}$ such that

1. $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ *for any* $a, b \in \mathcal{H}$,
2. $\mathrm{tr}(1) = 1$,
3. *for* $x \in \mathcal{H}_n \subset \mathcal{H}$ *we have*

$$\mathrm{tr}(xg_n) = z\,\mathrm{tr}(x).$$

- Relations 1.–3. suffice to compute tr of any word on the generators (g_i). For example:

$$\begin{aligned}
\mathrm{tr}(g_2g_1g_3g_2) &= \mathrm{tr}(g_2^2g_1g_3) = z\,\mathrm{tr}(g_2^2g_1) = z\,\mathrm{tr}\big(((q-1)g_2+q)g_1\big) \\
&= z(q-1)\mathrm{tr}(g_2g_1) + zq\,\mathrm{tr}(g_1) \\
&= z^2(q-1)\mathrm{tr}(g_1) + z^2q\,\mathrm{tr}(1) \\
&=
\end{aligned}$$

Ocneanu's functional

Theorem

For any $z \in \mathbb{C}$ *there exists a unique linear functional* $\mathrm{tr} : \mathcal{H} \to \mathbb{C}$ *such that*

1. $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ *for any* $a, b \in \mathcal{H}$,
2. $\mathrm{tr}(1) = 1$,
3. *for* $x \in \mathcal{H}_n \subset \mathcal{H}$ *we have*

$$\mathrm{tr}(xg_n) = z\,\mathrm{tr}(x).$$

- Relations 1.–3. suffice to compute tr of any word on the generators (g_i). For example:

$$\begin{aligned}
\mathrm{tr}(g_2g_1g_3g_2) &= \mathrm{tr}(g_2^2g_1g_3) = z\,\mathrm{tr}(g_2^2g_1) = z\,\mathrm{tr}\big(((q-1)g_2+q)g_1\big) \\
&= z(q-1)\mathrm{tr}(g_2g_1) + zq\,\mathrm{tr}(g_1) \\
&= z^2(q-1)\mathrm{tr}(g_1) + z^2q\,\mathrm{tr}(1) \\
&= z^3(q-1) + z^2q.
\end{aligned}$$

The first invariant

The first invariant

Theorem

- *Let L be a link,*

THE FIRST INVARIANT

THEOREM

- *Let L be a link,*
- *let* $\alpha \in B_n$ *be such that* $L = \widehat{\alpha}$*,*

THE FIRST INVARIANT

THEOREM

- *Let L be a link,*
- *let* $\alpha \in B_n$ *be such that* $L = \widehat{\alpha}$*,*
- *let e be the sum of exponents in* α *as a word on* $\sigma_1, \ldots, \sigma_{n-1}$*,*

The first invariant

Theorem

- *Let L be a link,*
- *let* $\alpha \in B_n$ *be such that* $L = \widehat{\alpha}$*,*
- *let e be the sum of exponents in* α *as a word on* $\sigma_1, \dots, \sigma_{n-1}$,
- *let*

$$\lambda = \frac{1-q+z}{qz},$$

The first invariant

Theorem

- *Let L be a link,*
- *let* $\alpha \in B_n$ *be such that* $L = \widehat{\alpha}$*,*
- *let e be the sum of exponents in* α *as a word on* $\sigma_1, \ldots, \sigma_{n-1}$,
- *let*
$$\lambda = \frac{1-q+z}{qz},$$
- *let*
$$X_L(q,\lambda) = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1} \left(\sqrt{\lambda}\right)^e \operatorname{tr}\bigl(\pi(\alpha)\bigr).$$

The first invariant

Theorem

- *Let L be a link,*
- *let* $\alpha \in B_n$ *be such that* $L = \widehat{\alpha}$*,*
- *let e be the sum of exponents in* α *as a word on* $\sigma_1, \dots, \sigma_{n-1}$*,*
- *let*
$$\lambda = \frac{1-q+z}{qz},$$
- *let*
$$X_L(q,\lambda) = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1} \left(\sqrt{\lambda}\right)^e \mathrm{tr}\big(\pi(\alpha)\big).$$

Then X_L *depends only on the equivalence class of L.*

The two-variable link polynomial

The two-variable link polynomial

Theorem
There exists a function

The two-variable link polynomial

Theorem

There exists a function

$$\left\{ \begin{array}{c} \textit{equivalence classes of} \\ \textit{oriented links} \end{array} \right\} \ni [L] \longmapsto P_L \in \mathbb{C}[t, t^{-1}, x, x^{-1}]$$

THEOREM

There exists a function

$$\left\{\begin{array}{c}\textit{equivalence classes of}\\ \textit{oriented links}\end{array}\right\} \ni [L] \longmapsto P_L \in \mathbb{C}[t, t^{-1}, x, x^{-1}]$$

such that if x and t satisfy

$$\begin{aligned} x &= \sqrt{\lambda}\sqrt{q}, \\ t &= \sqrt{q} - \frac{1}{\sqrt{q}} \end{aligned}$$

The two-variable link polynomial

Theorem

There exists a function

$$\left\{ \begin{array}{c} \textit{equivalence classes of} \\ \textit{oriented links} \end{array} \right\} \ni [L] \longmapsto P_L \in \mathbb{C}[t, t^{-1}, x, x^{-1}]$$

such that if x and t satisfy

$$x = \sqrt{\lambda}\sqrt{q},$$
$$t = \sqrt{q} - \frac{1}{\sqrt{q}}$$

then

$$P_L(x, t) = X_L(q, \lambda).$$

The skein relation

The skein relation

Theorem

Let L_+, L_- and L_0 be links whose diagrams are identical except for one crossing where they differ by

The skein relation

Theorem

Let L_+, L_- and L_0 be links whose diagrams are identical except for one crossing where they differ by

The skein relation

Theorem

Let L_+, L_- and L_0 be links whose diagrams are identical except for one crossing where they differ by

Let P_{L_+}, P_{L_-} and P_{L_0} be the corresponding polynomials.

The skein relation

Theorem

Let L_+, L_- and L_0 be links whose diagrams are identical except for one crossing where they differ by

Let P_{L_+}, P_{L_-} and P_{L_0} be the corresponding polynomials. Then we have

$$t^{-1}P_{L_+} - tP_{L_-} = xP_{L_0}.$$

The skein relation

Theorem

Let L_+, L_- and L_0 be links whose diagrams are identical except for one crossing where they differ by

Let P_{L_+}, P_{L_-} and P_{L_0} be the corresponding polynomials. Then we have

$$t^{-1}P_{L_+} - tP_{L_-} = xP_{L_0}.$$

- It follows that $P_L(x,t)$ is uniquely determined by its value on the trivial knot:

$$P_{\text{trivial knot}}(x,t) = 1.$$

Consequences of the skein relation

Consequences of the skein relation

- L — link,

Consequences of the skein relation

- L — link, L' — the same link with all orientations reversed.

Consequences of the skein relation

- L — link, L' — the same link with all orientations reversed. Then

$$P_{L'}(x,t) = P_L(x,t).$$

Consequences of the skein relation

- L — link, L' — the same link with all orientations reversed. Then
$$P_{L'}(x,t) = P_L(x,t).$$
- L — link,

Consequences of the skein relation

- L — link, L' — the same link with all orientations reversed. Then

$$P_{L'}(x,t) = P_L(x,t).$$

- L — link, $\widetilde{L}$ — mirror image of L.

Consequences of the skein relation

- L — link, L' — the same link with all orientations reversed. Then

$$P_{L'}(x,t) = P_L(x,t).$$

- L — link, $\widetilde{L}$ — mirror image of L. Then

$$P_{\widetilde{L}}(x,t) = P_L(t^{-1}, -x).$$

Consequences of the skein relation

- L — link, L' — the same link with all orientations reversed. Then

$$P_{L'}(x,t) = P_L(x,t).$$

- L — link, $\widetilde{L}$ — mirror image of L. Then

$$P_{\widetilde{L}}(x,t) = P_L(t^{-1}, -x).$$

- For

$L =$ 

Consequences of the skein relation

- L — link, L' — the same link with all orientations reversed. Then

$$P_{L'}(x,t) = P_L(x,t).$$

- L — link, $\widetilde{L}$ — mirror image of L. Then

$$P_{\widetilde{L}}(x,t) = P_L(t^{-1}, -x).$$

- For

$L =$ 

we have

$$P_L(x,t) = t^2x^2 - t^4 + 2t^2.$$

Consequences of the skein relation

- L — link, L' — the same link with all orientations reversed. Then

$$P_{L'}(x,t) = P_L(x,t).$$

- L — link, $\widetilde{L}$ — mirror image of L. Then

$$P_{\widetilde{L}}(x,t) = P_L(t^{-1}, -x).$$

- For

$L =$ 

  we have

$$P_L(x,t) = t^2x^2 - t^4 + 2t^2.$$

- Corollary: $L \not\approx \widetilde{L}$.