Compact quantum group actions on discrete quantum spaces and quantum Clifford theory

Special Session on Quantum Groups
AMS/MAA Joint Meeting, Atlanta

Piotr M. Sołtan
(joint work with K. De Commer, P. Kasprzak, & A. Skalski)

Department of Mathematical Methods in Physics
Faculty of Physics, University of Warsaw

January 6, 2017

Definition

Let M be a von Neumann algebra and let $\mathbb{G}$ be a compact quantum group. An **action** of $\mathbb{G}$ on M is an injective normal unital $*$-homomorphism $\alpha\colon\mathsf{M}\longrightarrow\mathsf{M}\bar{\otimes}L^{\infty}(\mathbb{G})$ such that

$$(\alpha\otimes\mathrm{id})\circ\alpha=(\mathrm{id}\otimes\Delta_{\mathbb{G}})\circ\alpha.$$

- We will only consider M of the form $\mathsf{M}=\prod\limits_{i\in I}\mathsf{M}_i$.
- Let p_i be the unit of M_i considered as a projection in M.
- $\boldsymbol{p}_i\colon\mathsf{M}\longrightarrow\mathsf{M}_i$ will be the canonical surjection.

Definition

For $i,j\in I$ we say that i and j are α-**related** (writing $i\sim_{\alpha}j$) if

$$\exists\,x\in\mathsf{M}_i\ (\boldsymbol{p}_j\otimes\mathrm{id})\big(\alpha(x)\big)\neq 0.$$

- For $i,j \in I$ define $\alpha_{j,i} : \mathsf{M}_i \longrightarrow \mathsf{M}_j \bar{\otimes} L^\infty(\mathbb{G})$ by

$$\alpha_{j,i}(x) = (\boldsymbol{p}_j \otimes \mathrm{id})\bigl(\alpha(x)\bigr), \qquad x \in \mathsf{M}_i.$$

Fact

The following are equivalent:

1. $i \sim_\alpha j$,
2. $\alpha_{j,i} \neq 0$,
3. $\alpha_{j,i}(\mathbb{1}_{\mathsf{M}_i}) \neq 0$.

- A more useful criterion for $i \sim_\alpha j$ can be given when α is **implemented**.

Definition

An action $\alpha:\mathsf{M}\longrightarrow\mathsf{M}\bar{\otimes}L^\infty(\mathbb{G})$ is **implemented** if there exist

- a Hilbert space $\mathscr{H}$,
- a faithful representation π of M on $\mathscr{H}$,
- a representation U of $\mathbb{G}$ on $\mathscr{H}$

such that

$$(\pi\otimes\mathrm{id})\big(\alpha(y)\big)=U\big(\mathbb{1}\otimes\pi(y)\big)U^*,\qquad y\in\mathsf{M}.$$

Fact

Any action can be implemented. For example:

- choose a faithful representation π_0 of M on $\mathscr{H}_0$,
- put $\mathscr{H}=\mathscr{H}_0\otimes L^2(\mathbb{G})$, $\pi=(\pi_0\otimes\mathrm{id})\circ\alpha$,
- let $U=\mathrm{W}^{\mathbb{G}}_{23}\in\mathrm{B}(\mathscr{H}_0)\bar{\otimes}\ell^\infty(\widehat{\mathbb{G}})\bar{\otimes}L^\infty(\mathbb{G})\subset\mathrm{B}(\mathscr{H})\bar{\otimes}L^\infty(\mathbb{G})$.

- Assume $\alpha : \mathsf{M} \longrightarrow \mathsf{M} \bar{\otimes} L^\infty(\mathbb{G})$ is implemented:

$$(\pi \otimes \mathrm{id})\bigl(\alpha(y)\bigr) = U\bigl(\mathbb{1} \otimes \pi(y)\bigr)U^*, \qquad y \in \mathsf{M}.$$

- Since $\mathsf{M} = \prod\limits_{i \in I} \mathsf{M}_i$, we have
  - $\mathscr{H} = \bigoplus\limits_{i \in I} \mathscr{H}_i$, where $\mathscr{H}_i = \pi(p_i)\mathscr{H}$ and
  - for each $y \in \mathsf{M}$

$$\pi(y) = \bigoplus_{i \in I} \pi_i(y_i),$$

  with π_i a faithful representation of M_i on $\mathscr{H}_i$ and $y_i = \boldsymbol{p}_i(y)$.
- For $k, l \in I$ put $U_{k,l} = \bigl(\pi(p_k) \otimes \mathbb{1}\bigr)U\bigl(\pi(p_l) \otimes \mathbb{1}\bigr)$. Then for each i,j we have

$$(\pi_j \otimes \mathrm{id})\bigl(\alpha_{j,i}(x)\bigr) = U_{j,i}\bigl(\pi_i(x) \otimes \mathbb{1}\bigr)U_{j,i}^*, \qquad x \in \mathsf{M}_i$$

  because U implements α.
- Clearly $(\pi_j \otimes \mathrm{id})\bigl(\alpha_{j,i}(\mathbb{1}_{\mathsf{M}_i})\bigr) = U_{j,i}U_{j,i}^*$.

- Assume $\alpha : \mathsf{M} \to \mathsf{M} \bar{\otimes} L^{\infty}(\mathbb{G})$ is implemented by U.
  - For all i,j we have $(\pi_j \otimes \mathrm{id})\big(\alpha_{j,i}(\mathbb{1}_{\mathsf{M}_i})\big) = U_{j,i}U_{j,i}^*$ and
  - π_j is faithful on M_j.
- Thus we get

COROLLARY

We have

$$\Big(i \sim_\alpha j \Big) \Longleftrightarrow \Big(U_{j,i} \neq 0 \Big).$$

- This will help in studying properties of $\sim_\alpha$

PROPOSITION

The relation $\sim_\alpha$ is symmetric.

PROOF:

Assume α is implemented by $U \in \mathrm{B}(\mathscr{H}) \mathbin{\bar{\otimes}} L^\infty(\mathbb{G})$. We have

$$\Big(i \sim_\alpha j \Big) \Longleftrightarrow \Big(U_{j,i} \neq 0 \Big)$$

and this is equivalent to

$$\exists\, \xi \in \mathscr{H}_j,\ \eta \in \mathscr{H}_i\ (\omega_{\xi,\eta} \otimes \mathrm{id})(U) \neq 0.$$

Now $(\omega_{\xi,\eta} \otimes \mathrm{id})(U) \in \mathrm{D}(S)$ and

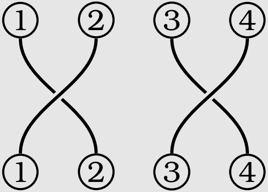
$$0 \neq S\big((\omega_{\xi,\eta} \otimes \mathrm{id})(U)\big) = (\omega_{\xi,\eta} \otimes \mathrm{id})(U^*).$$

This means that $U^*_{i,j} = (U^*)_{j,i} \neq 0$, so $U_{i,j} \neq 0$ and $j \sim_\alpha i$. $\square$

REMARK

The relation $\sim_\alpha$ need not be transitive. For example take

- $\mathsf{M} = L^\infty(\{1,2,3,4\}) = L^\infty(\{1\}) \oplus L^\infty(\{2,3\}) \oplus L^\infty(\{4\})$
- $\mathbb{G} = \mathbb{Z}_2$ acting by



- Then $\{1\} \sim_\alpha \{2,3\}$ and $\{2,3\} \sim_\alpha \{4\}$, but $\{1\} \not\sim_\alpha \{4\}$.

Proposition

If for each $i \in I$ the algebra M_i is a factor then $\sim_\alpha$ is an equivalence relation.

Proof:

From $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta_{\mathbb{G}}) \circ \alpha$ it follows that

$$(\mathrm{id} \otimes \Delta_{\mathbb{G}})\big(\alpha_{j,i}(x)\big) = \sum_{k \in I} (\alpha_{j,k} \otimes \mathrm{id})\big(\alpha_{k,i}(x)\big), \qquad i,j \in I,\ x \in \mathsf{M}_i.$$

Moreover, since each M_i is a factor and α is normal, we have

$$\Big(\ k \sim_\alpha l\ \Big) \Longleftrightarrow \Big(\ \ker \alpha_{k,l} = \{0\}\ \Big)$$

for any $k, l \in I$.

Proof (contd.):

Assume $i \sim_\alpha l$ and $l \sim_\alpha j$. Then

$$\begin{aligned}(\mathrm{id} \otimes \Delta_{\mathbb{G}})\big(\alpha_{j,i}(\mathbb{1}_{\mathsf{M}_i})\big) &= \sum_{k \in I} (\alpha_{j,k} \otimes \mathrm{id})\big(\alpha_{k,i}(\mathbb{1}_{\mathsf{M}_i})\big) \\ &\geq (\alpha_{j,l} \otimes \mathrm{id})\big(\alpha_{l,i}(\mathbb{1}_{\mathsf{M}_i})\big) \neq 0.\end{aligned}$$

Since the above sum is a sum of orthogonal projections, we obtain $(\mathrm{id} \otimes \Delta_{\mathbb{G}})\big(\alpha_{j,i}(\mathbb{1}_{\mathsf{M}_i})\big) \neq 0$, so $\alpha_{j,i}(\mathbb{1}_{\mathsf{M}_i}) \neq 0$ and thus $i \sim_\alpha j$. Now for each i there exists j such that $i \sim_\alpha j$ (otherwise $\ker \alpha \neq \{0\}$). This implies reflexivity of $\sim_\alpha$. □

- We will call equivalence classes of $\sim_\alpha$ **orbits** of the action α.

Remark

Let $A \subset I$ be an equivalence class of $\sim_\alpha$. Then $\alpha : \mathsf{M} \longrightarrow \mathsf{M} \bar{\otimes} L^\infty(\mathbb{G})$ restricts to $\prod_{i \in A} \mathsf{M}_i$ giving an action of $\mathbb{G}$ on $\prod_{i \in A} \mathsf{M}_i$. In particular setting

$$\mathbb{1}_A = \sum_{i \in A} p_i$$

we get an **invariant element**:

$$\alpha(\mathbb{1}_A) = \mathbb{1}_A \otimes \mathbb{1}.$$

Corollary

If α is ergodic then $\sim_\alpha$ is a total relation.

Theorem

Let β be an ergodic action of a compact quantum group on a von Neumann algebra $\mathsf{N} = M_n(\mathbb{C}) \oplus \widetilde{\mathsf{N}}$. Then $\dim \mathsf{N} < +\infty$.

Corollary

Let α be an action of a compact quantum group on $\mathsf{M} = \prod_{i \in I} M_{n_i}(\mathbb{C})$. Then all orbits of α (equivalence classes of $\sim_\alpha$) are finite.

Steps of proof:

- Restrict to one class: assume $i \sim_\alpha j$ for all i, j,
- take p – a minimal projection in $\{m \in \mathsf{M} \,|\, \alpha(m) = m \otimes \mathbb{1}\}$,
- α restricts to an action on $p\mathsf{M}p$ which is ergodic,

STEPS OF PROOF (CONTD.):

- $p\mathsf{M}p$ is itself a product of matrix algebras, so by Theorem, $\dim p\mathsf{M}p < +\infty$,
- it follows that $I_p = \{i \in I \,|\, p_i p \neq 0\}$ is finite,
- take $i \in I_p$ and $j \in I \setminus I_p$; we have $p_i p \neq 0$, but

$$\begin{aligned}\alpha_{j,i}(p_i p) = (\boldsymbol{p}_j \otimes \mathrm{id})\bigl(\alpha(p_i p)\bigr) &\leq (\boldsymbol{p}_j \otimes \mathrm{id})\bigl(\alpha(p)\bigr)\\ &= (\boldsymbol{p}_j \otimes \mathrm{id})(p \otimes \mathbb{1}) = \boldsymbol{p}_j(p) \otimes \mathbb{1} = 0\end{aligned}$$

  (because $p_j p = 0$),
- however, we have $i \sim_\alpha j$, so $\ker \alpha_{j,i} \neq \{0\}$ and hence we arrive at a contradiction. □

- Let $\mathbb{\Gamma}$ be a discrete quantum group:

$$\ell^{\infty}(\mathbb{\Gamma}) = \prod_{\gamma \in \operatorname{Irr} \widehat{\mathbb{\Gamma}}} M_{n_\gamma}(\mathbb{C}).$$

- and let $\mathbb{\Lambda}$ be a quantum subgroup of $\mathbb{\Gamma}$
  - $L^{\infty}(\widehat{\mathbb{\Lambda}}) \subset L^{\infty}(\widehat{\mathbb{\Gamma}})$, ($\mathbb{\Lambda}$ is closed)
  - $\boldsymbol{\pi} : \ell^{\infty}(\mathbb{\Gamma}) \longrightarrow \ell^{\infty}(\mathbb{\Lambda})$. ($\mathbb{\Lambda}$ is open)
- Put $\ell^{\infty}(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) = \left\{x \in \ell^{\infty}(\mathbb{\Gamma}) \,\middle|\, (\boldsymbol{\pi} \otimes \mathrm{id})\Delta_{\mathbb{\Gamma}}(x) = \mathbb{1} \otimes x\right\}$.
- Let $\mathbb{G} = \widehat{\mathbb{\Gamma}}$. We have

$$\mathrm{W}^{\mathbb{G}}\big(\ell^{\infty}(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \otimes \mathbb{1}\big)\mathrm{W}^{\mathbb{G}*} \subset \ell^{\infty}(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \,\bar{\otimes}\, L^{\infty}(\widehat{\mathbb{\Gamma}})$$

  which yields an action of $\mathbb{G}$ on $\ell^{\infty}(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$:

$$\alpha(x) = \mathrm{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{G}*}, \qquad x \in \ell^{\infty}(\mathbb{\Lambda}\backslash\mathbb{\Gamma}).$$

EXAMPLE

Consider a special case:

- let $\mathbb{H} \subset \mathbb{G}$ be a normal closed quantum subgroup,
- let $\mathbb{\Gamma} = \widehat{\mathbb{G}}$ and $\mathbb{\Lambda} = \widehat{\mathbb{G}/\mathbb{H}}$.

Then $\mathbb{G}$ acts on $\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) = \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) = \ell^\infty(\widehat{\mathbb{H}})$.

- $\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ is a product of matrix algebras:

$$\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) = \prod_{i \in I} \mathsf{M}_i$$

  with each $\mathsf{M}_i = M_{m_i}(\mathbb{C})$.
- The action of $\mathbb{G} = \widehat{\mathbb{\Gamma}}$ on $\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ defines the equivalence relation $\sim_\alpha$ on I.

- Denote by $\mathbb{1}_j$ the unit of $\mathsf{M}_j \subset \ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ viewed as a projection in $\ell^\infty(\mathbb{\Gamma})$.

THEOREM

For any $i \in I$ the element

$$\sum_{j \sim_\alpha i} \mathbb{1}_j \in \ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \subset \ell^\infty(\mathbb{\Gamma})$$

is the central support $z(\mathbb{1}_i)$ in $\ell^\infty(\mathbb{\Gamma})$ of the projection $\mathbb{1}_i$. Moreover $z(\mathbb{1}_i)$ is orthogonal to $z(\mathbb{1}_j)$ if i is not equivalent to j.

- In particular for any $\kappa \in \operatorname{Irr}\widehat{\mathbb{\Gamma}}$ there exists $i \in I$ such that
  1. for all $j \in I$ we have $p_\kappa \mathbb{1}_j \neq 0$ if and only if $j \sim_\alpha i$,
  2. we have $p_\kappa\Bigl(\sum_{j \sim_\alpha i} \mathbb{1}_j\Bigr) = p_\kappa$.

Example revisited

- When $\mathbb{\Lambda} = \widehat{\mathbb{G}/\mathbb{H}}$ for a closed normal subgroup $\mathbb{H}$ of $\mathbb{G}$, the theorem says that for an irrep κ of $\mathbb{G}$ (or $\ell^{\infty}(\widehat{\mathbb{G}})$) the restriction of κ to $\mathbb{H}$ (or $\ell^{\infty}(\widehat{\mathbb{H}})$) is a direct sum of irreps of $\mathbb{H}$ constituting one class of the equivalence relation $\sim_{\alpha}$ on $I = \operatorname{Irr}\mathbb{H}$.
- for classical groups G and H the irreps of H in one orbit of the action of G (by conjugation) all have the same dimension.

Theorem

Let $\mathbb{G}$ be a compact quantum group of Kac type and let $\mathbb{H}$ be a closed normal quantum subgroup of $\mathbb{G}$. Then any two irreducible representations σ and τ of $\mathbb{H}$ in the same orbit have the same dimension. Moreover, if π is any irreducible representation of $\mathbb{G}$ with $\pi(\mathbb{1}_\sigma) \neq 0$, then also the multiplicity of σ in π is the same as the multiplicity of τ in π.

THEOREM

Consider the following three conditions

1. $\widehat{\mathbb{G}}$ *is **torsion free**,*
2. $\mathbb{G}$ *is satisfies the* (TO)*-**condition**,*
3. $\mathbb{G}$ *is **connected**.*

All actions of $\mathbb{G}$ on finite dimensional C*-*algebras are direct sums of actions Morita equivalent to trivial action on* $\mathbb{C}$

For any action of $\mathbb{G}$ on a product of matrix algebras the orbits are trivial

There is no finite quantum group $\mathbb{H}$ such that $\mathrm{Pol}(\mathbb{H}) \subset \mathrm{Pol}(\mathbb{G})$ *as a Hopf* $*$*-subalgebra*

Then

$$① \implies ② \implies ③.$$

In general neither of the implications can be reversed.

Definition

Let $\mathbb{\Lambda}$ be a quantum subgroup of a discrete quantum group $\mathbb{\Gamma}$. For $\sigma, \tau \in \operatorname{Irr}\widehat{\mathbb{\Gamma}}$ we say that σ and τ are $\mathbb{\Lambda}$-**related** if there exists $\gamma \in \operatorname{Irr}\widehat{\mathbb{\Lambda}}$ such that $\tau \subset \sigma \otimes \gamma$.

- Recall that in this situation we have an action α of $\mathbb{G}$ on $\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) = \prod\limits_{i \in I} \mathsf{M}_i$.
- For $i \in I$ define $\mathbb{\Gamma}\text{-supp}(\mathbb{1}_i) = \left\{\kappa \in \operatorname{Irr}\widehat{\mathbb{\Gamma}} \,\middle|\, p_\kappa \mathbb{1}_i \neq 0\right\}$.

Theorem

1. *For $i, j \in I$ we have $i \sim_\alpha j$ iff $\mathbb{\Gamma}\text{-supp}(\mathbb{1}_i) = \mathbb{\Gamma}\text{-supp}(\mathbb{1}_j)$,*
2. *two elements $\sigma, \tau \in \operatorname{Irr}\widehat{\mathbb{\Gamma}}$ are $\mathbb{\Lambda}$-related iff there exists $i \in I$ such that $\sigma, \tau \in \mathbb{\Gamma}\text{-supp}(\mathbb{1}_i)$.*