An application of property (T) for discrete quantum groups

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COMPACT QUANTUM GROUPS

Definition

$$\mathbb{G} = \big(\mathrm{C}(\mathbb{G}), \Delta\big)$$

- $\mathrm{C}(\mathbb{G})$ — unital C^*-algebra
- $\Delta\colon\ \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$

$$\begin{array}{ccc} \mathrm{C}(\mathbb{G}) & \xrightarrow{\quad\Delta\quad} & \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G}) \\ {\scriptstyle\Delta}\big\downarrow & & \big\downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G}) & \xrightarrow[\mathrm{id}\otimes\Delta]{} & \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G}) \end{array}$$

- $\Delta\big(\mathrm{C}(\mathbb{G})\big)\big(\mathbf{1} \otimes \mathrm{C}(\mathbb{G})\big) = \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$
- $\big(\mathrm{C}(\mathbb{G}) \otimes \mathbf{1}\big)\Delta\big(\mathrm{C}(\mathbb{G})\big) = \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$

Examples

- G — compact group,
  - $\mathrm{C}(\mathbb{G}) := \mathrm{C}(G)$
  - $\Delta(f)(x, y) = f(xy)$
- Γ – discrete group
  - $\mathrm{C}(\mathbb{G}) := \mathrm{C}^*(\Gamma)$
  - $\Delta(\gamma) = \gamma \otimes \gamma$

  or
  - $\mathrm{C}(\mathbb{G}) := \mathrm{C}^*_{\mathrm{r}}(\Gamma)$
  - $\Delta(\gamma) = \gamma \otimes \gamma$

The Hopf algebra

Theorem (S.L. Woronowicz)

Let $\mathbb{G}$ be a compact quantum group. There exists a unique dense Hopf $*$-subalgebra $\mathrm{Pol}(\mathbb{G}) \subset \mathrm{C}(\mathbb{G})$.

- $\mathrm{Pol}(\mathbb{G})$ is a **Hopf algebra**, so
  - $\mathrm{Pol}(\mathbb{G})$ is a unital $*$-subalgebra of $\mathrm{C}(\mathbb{G})$,
  - $\Delta\big(\mathrm{Pol}(\mathbb{G})\big) \subset \mathrm{Pol}(\mathbb{G}) \odot \mathrm{Pol}(\mathbb{G})$,
  - there is a counit (denoted ϵ) and an antipode on $\mathrm{Pol}(\mathbb{G})$.
- Moreover
  - for $\mathbb{G}$ classical, i.e. $\mathrm{C}(\mathbb{G}) = \mathrm{C}(G)$, the subalgebra $\mathrm{Pol}(\mathbb{G})$ is the algebra of **regular functions** on G,
  - if $\mathrm{C}(\mathbb{G}) = \mathrm{C}^*(\Gamma)$ (or $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$) we have $\mathrm{Pol}(\mathbb{G}) = \mathbb{C}[\Gamma]$.
- $\mathrm{Pol}(\mathbb{G})$ is the linear span of matrix elements of irreducible corepresentations of $\mathbb{G}$.

Norms on $\mathrm{Pol}(\mathbb{G})$

- maximal (universal) C*-norm
  $\rightsquigarrow$ the completion: $C(\mathbb{G}_{\max})$
- minimal (reduced) C*-norm
  $\rightsquigarrow$ the completion: $C(\mathbb{G}_{\min})$
- $\|a\|_{\sim} = \max\{\|a\|, |\epsilon(a)|\}$
  $\rightsquigarrow$ the completion: $C(\widetilde{\mathbb{G}})$

Example: $\mathrm{Pol}(\mathbb{G}) = \mathbb{C}[\Gamma]$

$\rightsquigarrow C(\mathbb{G}_{\max}) = C^*_{\mathrm{full}}(\Gamma)$

$\rightsquigarrow C(\mathbb{G}_{\min}) = C^*_{\mathrm{r}}(\Gamma)$

$\rightsquigarrow C(\widetilde{\mathbb{G}}) = ??$

Definition

A C^**-norm on* $\mathrm{Pol}(\mathbb{G})$ *is a* ***quantum group norm*** *if*

$$\Delta\colon \mathrm{Pol}(\mathbb{G}) \longrightarrow \mathrm{Pol}(\mathbb{G}) \otimes \mathrm{Pol}(\mathbb{G})$$

extends to completions.

Fact

All of the above C^**-norms are quantum group norms.*

Exotic completions

- We are interested in quantum group norms **quantum group norms** on $\mathrm{Pol}(\mathbb{G})$ such that if $\mathrm{C}(\mathbb{G})$ is the completion we have
  - $\mathrm{C}(\mathbb{G}_{\mathrm{min}}) \neq \mathrm{C}(\mathbb{G})$,
  - $\mathrm{C}(\mathbb{G}) \neq \mathrm{C}(\mathbb{G}_{\mathrm{max}})$,
  - $\mathrm{C}(\mathbb{G}) \neq \mathrm{C}(\widetilde{\mathbb{G}}) \neq \mathrm{C}(\mathbb{G}_{\mathrm{max}})$

  (in the sense that the canonical epimorphisms are not isomorphisms).
- Another interesting possibility is
  - $\mathrm{C}(\mathbb{G}) \neq \mathrm{C}(\widetilde{\mathbb{G}}) = \mathrm{C}(\mathbb{G}_{\mathrm{max}})$.
- We call such norms **exotic** quantum group norms.
- Existence of exotic norms is interesting for the theory of quantum group actions.

Discrete quantum groups

- Each compact quantum group $\mathbb{G}$ comes with its **discrete dual**
$$\widehat{\mathbb{G}} = \left(c_0(\widehat{\mathbb{G}}), \widehat{\Delta}\right).$$
- Crucial fact: $c_0(\widehat{\mathbb{G}})$ is a direct sum of matrix algebras.
- If $\mathbb{G}$ is classical ($C(\mathbb{G}) = C(G)$) and abelian then
$$c_0(\widehat{\mathbb{G}}) = c_0(\widehat{G}) = \bigoplus_{\widehat{G}} \mathbb{C}$$
- Representations of the C*-algebra $c_0(\widehat{\mathbb{G}})$ are in natural bijection with corepresentations of $\mathbb{G}$.
- Representations of the C*-algebra $C(\mathbb{G}_{\max})$ are in natural bijection with corepresentations of $\widehat{\mathbb{G}}$.
- In 2008 Pierre Fima defined property (T) for discrete quantum groups. The analog of a finite set in $\widehat{\mathbb{G}}$ is a finite sum of simple summands of $c_0(\widehat{\mathbb{G}})$.

Examples

1. Let $\mathbb{G}$ be classical: $\mathrm{C}(\mathbb{G}) = \mathrm{C}(G)$, where G is a compact group. Then
   - we have
   $$c_0(\widehat{\mathbb{G}}) = \bigoplus_{\pi\text{ – irrep of } G} M_{\dim\pi}(\mathbb{C}),$$
   - $\widehat{\Delta}$ reflects the tensor product of representations of G.
2. Let Γ be a discrete group and $\mathbb{G} = \big(\mathrm{C}^*(\Gamma), \Delta\big)$. Then
   - $c_0(\widehat{\mathbb{G}}) = c_0(\Gamma)$,
   - $\widehat{\Delta}\colon\ c_0(\widehat{\mathbb{G}}) \to \mathrm{M}\big(c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{G}})\big)$
   $$\widehat{\Delta}(f)(x, y) = f(xy).$$
   - $\widehat{\Delta}$ is a **morphism** $c_0(\widehat{\mathbb{G}}) \to c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{G}})$.
   - $\widehat{\mathbb{G}} = (c_0(\widehat{\mathbb{G}}), \widehat{\Delta})$ is a discrete quantum group.
   - $\widehat{\mathbb{G}}$ has property (T) in the sense of Fima if and only if Γ has property (T).

Other characterizations

Theorem (David Kyed & P.M.S.)

The following are equivalent:

- $\widehat{\mathbb{G}}$ *has property* (T) *in the sense of Fima,*
- *the counit* ϵ *is an isolated point of* $\mathrm{Spec}\big(\mathrm{C}(\mathbb{G}_{\max})\big)$,
- *all finite dimensional representations are isolated points of* $\mathrm{Spec}\big(\mathrm{C}(\mathbb{G}_{\max})\big)$,
- *the* C^**-algebra* $\mathrm{C}(\mathbb{G}_{\max})$ *has property* (T) *of Bekka,*
- *there exists a unique minimal projection* p *in the center of* $\mathrm{C}(\mathbb{G}_{\max})$ *with* $\epsilon(p) = 1$,
- *there exists a minimal projection* $p \in \mathrm{C}(\mathbb{G}_{\max})$ *with* $\epsilon(p) = 1$,
- $\widehat{\mathbb{G}}$ *has property* (T) *as defined by Petrescu & Joita (1992, for Kac algebras only),*
- $\widehat{\mathbb{G}}$ *has property* (T) *as defined by Bédos, Conti & Tuset (2005, for algebraic quantum groups).*

First exotic examples

Theorem

Take a non-coamenable $\mathbb{G}$.* *Then*

- $\mathrm{C}(\mathbb{G}_{\mathrm{min}}) \neq \mathrm{C}(\widetilde{\mathbb{G}_{\mathrm{min}}})$,
- *if* $\mathrm{C}(\widetilde{\mathbb{G}_{\mathrm{min}}}) = \mathrm{C}(\mathbb{G}_{\mathrm{max}})$ *then* $\widehat{\mathbb{G}}$ *has property* (T).

This provides many examples such that

$$\mathbb{G}_{\mathrm{min}} \neq \mathbb{G} \neq \mathbb{G}_{\mathrm{max}}$$

(take $\mathbb{G} = \widetilde{\mathbb{G}_{\mathrm{min}}}$ with $\mathbb{G}$ without property (T)).

*i.e. $\mathrm{C}(\mathbb{G}_{\mathrm{min}}) \neq \mathrm{C}(\mathbb{G}_{\mathrm{max}})$

Special representation

- Let π be the representation of $\mathrm{C}(\mathbb{G}_{\max})$ which is the direct sum of all infinite-dimensional irreducible representations.

Theorem

If $\widehat{\mathbb{G}}$ has property (T) then the C^*-norm on $\mathrm{Pol}(\mathbb{G})$ defined by π is a quantum group norm.

- Denote the resulting quantum group by $\mathbb{G}_{\pi}$.

More exotic examples

- Take $\widehat{\mathbb{G}}$ — infinite property (T) discrete quantum group.
- $\mathbb{G}_{\boldsymbol{\pi}}$ does not admit a continuous counit, so

$$\mathbb{G}_{\boldsymbol{\pi}} \neq \widetilde{\mathbb{G}_{\boldsymbol{\pi}}}.$$

- It could happen that $\mathbb{G}_{\text{min}} = \mathbb{G}_{\boldsymbol{\pi}}$, but in most cases

$$\mathbb{G}_{\text{min}} \neq \mathbb{G}_{\boldsymbol{\pi}}.$$

- there are examples when $\widetilde{\mathbb{G}_{\boldsymbol{\pi}}} = \mathbb{G}_{\text{max}}$, but in most cases

$$\widetilde{\mathbb{G}_{\boldsymbol{\pi}}} \neq \mathbb{G}_{\text{max}}.$$

Summary

- $\mathbb{G}$ — coamenable

$$\mathbb{G}_{\mathrm{min}} = \mathbb{G} = \widetilde{\mathbb{G}} = \mathbb{G}_{\mathrm{max}}.$$

- $\mathbb{G}$ — non-coamenable, $\widehat{\mathbb{G}}$ not Kazhdan

$$\mathbb{G}_{\mathrm{min}} = \mathbb{G} \neq \widetilde{\mathbb{G}} \neq \mathbb{G}_{\mathrm{max}}.$$

- $\widehat{\mathbb{G}}$ — Kazhdan, minimally almost periodic

$$\mathbb{G}_{\mathrm{min}} \neq \mathbb{G} \neq \widetilde{\mathbb{G}} = \mathbb{G}_{\mathrm{max}}.$$

- $\widehat{\mathbb{G}}$ — Kazhdan, not minimally almost periodic

$$\mathbb{G}_{\mathrm{min}} \neq \mathbb{G} \neq \widetilde{\mathbb{G}} \neq \mathbb{G}_{\mathrm{max}}.$$

Thank you