Compact quantum groups defined by universal properties

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October 22, 2010

Lecture outline

Introduction

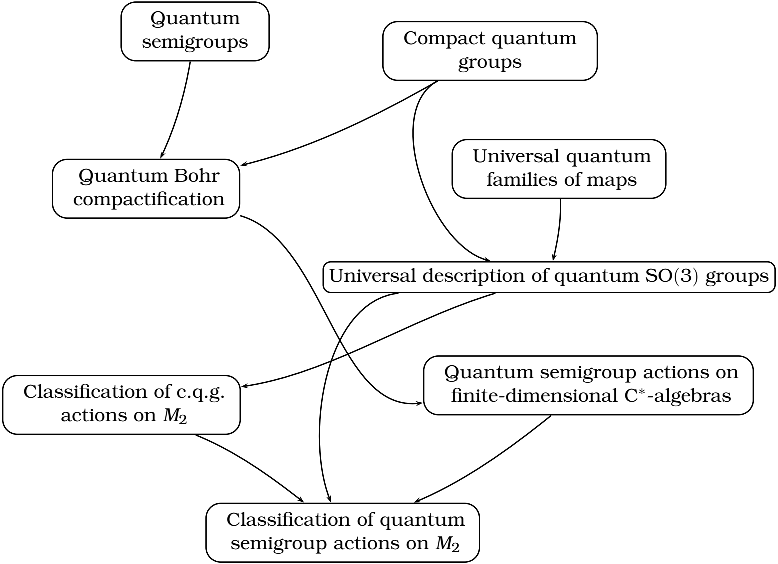
Quantum spaces

Quantum groups

Quantum group actions

Actions on $M_2(\mathbb{C})$

STRUCTURE OF HABILITATION RESEARCH



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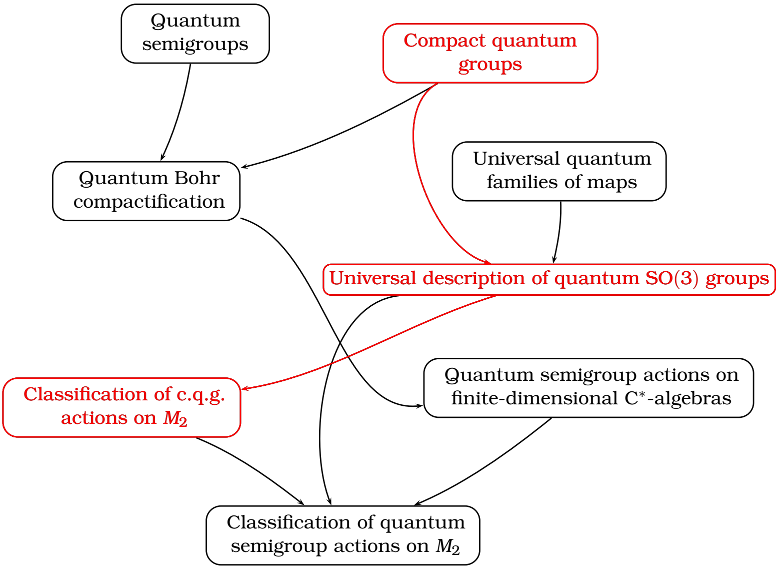

C*-algebras

C^*-algebras

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- Analogous fact is also true for locally compact spaces and algebras without unit.

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  - every theorem about quantum spaces is a theorem about C^*-algebras.

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- The morphism Δ is called the **comultiplication**.

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  This quantum group is called the **dual group** of Γ.

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- Δ_C is defined as:

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$$\Delta_C(c) = -q^2c\otimes k + l\otimes c - qg^*\otimes c^* + c\otimes(\mathbf{1}-k),$$

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Let $\mathbb{H} = (D, \Delta_D)$ be a compact quantum group and let $\beta \colon B \to B \otimes D$ be an action of $\mathbb{H}$ on B. Then there are

- $q \in]0, 1[$,
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