Invariants of quantum groups related to the scaling group

German-Polish Workshop on Quantum groups, graphs and symmetries

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The setup

- Let $\mathbb{G}$ be a locally compact quantum group and left Haar measure φ.
- Let $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ be the modular group of φ and let $\{\tau_t^\mathbb{G}\}_{t\in\mathbb{R}}$ denote the scaling group of $\mathbb{G}$.
- Let δ be the modular element of $\mathbb{G}$.
- Denote the group of inner automorphisms of the von Neumann algebra $L^\infty(\mathbb{G})$ by $\mathrm{Inn}(L^\infty(\mathbb{G}))$ and the group of approximately inner automorphisms of $L^\infty(\mathbb{G})$ by $\overline{\mathrm{Inn}}(L^\infty(\mathbb{G}))$.

The invariants

Definition

We define

$$
\begin{aligned}
T^{\tau}(\mathbb{G}) &= \bigl\{t \in \mathbb{R} \,\big|\, \tau^{\mathbb{G}}_t = \mathrm{id}\bigr\},\\
T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) &= \bigl\{t \in \mathbb{R} \,\big|\, \tau^{\mathbb{G}}_t \in \mathrm{Inn}\bigl(L^\infty(\mathbb{G})\bigr)\bigr\},\\
T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) &= \bigl\{t \in \mathbb{R} \,\big|\, \tau^{\mathbb{G}}_t \in \overline{\mathrm{Inn}}\bigl(L^\infty(\mathbb{G})\bigr)\bigr\},\\
T^{\sigma}(\mathbb{G}) &= \bigl\{t \in \mathbb{R} \,\big|\, \sigma^{\varphi}_t = \mathrm{id}\bigr\},\\
T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) &= \bigl\{t \in \mathbb{R} \,\big|\, \sigma^{\varphi}_t \in \mathrm{Inn}\bigl(L^\infty(\mathbb{G})\bigr)\bigr\},\\
T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) &= \bigl\{t \in \mathbb{R} \,\big|\, \sigma^{\varphi}_t \in \overline{\mathrm{Inn}}\bigl(L^\infty(\mathbb{G})\bigr)\bigr\},\\
\mathrm{Mod}(\mathbb{G}) &= \bigl\{t \in \mathbb{R} \,\big|\, \delta^{\mathrm{i}t} = \mathbb{1}\bigr\}.
\end{aligned}
$$

Some properties of the invariants

- The sets $T^{\circ}_{\bullet}(\mathbb{G})$ are subgroups of $\mathbb{R}$ and are isomorphism invariants of the quantum group $\mathbb{G}$.
- $T^{\tau}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}})$.
- $T^{\bullet}(\mathbb{G})$, $T^{\bullet}_{\overline{\mathrm{Inn}}}(\mathbb{G})$, and $\mathrm{Mod}(\mathbb{G})$ are closed.
- We would obtain the same groups $T^{\sigma}(\mathbb{G})$, $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G})$, and $T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G})$ if we chose the right Haar measure instead of the left one.
- $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G})$ is equal to the Connes' invariant $T(L^{\infty}(\mathbb{G}))$. Consequently, $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G})$ depends only on the von Neumann algebra $L^{\infty}(\mathbb{G})$. It is also the case for $T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G})$.

Some properties of the invariants

Proposition

For any locally compact quantum group $\mathbb{G}$ we have

$$\begin{aligned} T^{\sigma}(\mathbb{G}) &= T^{\tau}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}), \\ T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}) &= T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}), \\ T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}) &= T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}), \\ \operatorname{Mod}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}) &\subset \tfrac{1}{2}\, T^{\tau}(\mathbb{G}). \end{aligned}$$

- The first equality above together with $T^{\tau}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}})$ reduces the list to 11 (invariants $T^{\sigma}(\mathbb{G})$, $T^{\sigma}(\widehat{\mathbb{G}})$ and $T^{\tau}(\widehat{\mathbb{G}})$ are determined by the remaining ones).
- If $\mathbb{G}$ is compact then $\operatorname{Mod}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathbb{R}$.
- If additionally $L^{\infty}(\mathbb{G})$ is semifinite then $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = \mathbb{R}$.

Examples

Example: the quantum $E(2)$ group

With $\mathbb{G} = \mathrm{E}_q(2)$ for some $q \in \,]0, 1[$ we have

$$T^\tau(\mathbb{G}) = T^\tau_{\mathrm{Inn}}(\mathbb{G}) = T^\tau_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\sigma(\mathbb{G}) = T^\tau(\widehat{\mathbb{G}}) = T^\sigma(\widehat{\mathbb{G}}) = \mathrm{Mod}(\widehat{\mathbb{G}}) = \tfrac{\pi}{\log q}\mathbb{Z},$$

$$T^\sigma_{\mathrm{Inn}}(\mathbb{G}) = T^\sigma_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\tau_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^\tau_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^\sigma_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^\sigma_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathrm{Mod}(\mathbb{G}) = \mathbb{R}.$$

Examples

Example: quantum "$az+b$" groups

Let $\mathbb{G}$ be the quantum "$az+b$" group for the deformation parameter q in one of the three cases:

1. $q = \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}$ with $N = 6, 8, \dots,$
2. $q \in \,]0,1[,$
3. $q = \mathrm{e}^{1/\rho}$ with $\operatorname{Re}\rho < 0$, $\operatorname{Im}\rho = \frac{N}{2\pi}$ with $N = \pm 2, \pm 4, \dots$.

Then

$$T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = T^{\tau}_{\underline{\mathrm{Inn}}}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\tau}_{\underline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) = T^{\sigma}_{\underline{\mathrm{Inn}}}(\mathbb{G}) = T^{\sigma}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\underline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathbb{R},$$

$$T^{\tau}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}}) = T^{\sigma}(\mathbb{G}) = T^{\sigma}(\widehat{\mathbb{G}}) = \operatorname{Mod}(\mathbb{G}) = \operatorname{Mod}(\widehat{\mathbb{G}}) = \begin{cases} \{0\} & \text{in cases ① and ③} \\ \frac{\pi}{\log q}\mathbb{Z} & \text{in case ②} \end{cases}.$$

Examples

Example: U_F^+

Let $\mathbb{G}$ be the quantum group U_F^+. Then $L^\infty(\mathbb{G})$ is a full factor, so $\mathrm{Inn}(L^\infty(\mathbb{G})) = \overline{\mathrm{Inn}}(L^\infty(\mathbb{G}))$ (Vaes).

- $\mathbb{G}$ is compact, so $\mathrm{Mod}(\mathbb{G}) = T^\tau_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^\tau_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^\sigma_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^\sigma_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathbb{R}$.
- If $\mathbb{G}$ is not of Kac type ($\lambda F^*F \neq \mathbb{1}$) then

$$T^\tau_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\tau_{\mathrm{Inn}}(\mathbb{G}) = T^\tau(\mathbb{G}) = \bigcap_{\Lambda \in \mathrm{Sp}(F^*F \otimes (F^*F)^{-1}) \setminus \{1\}} \frac{2\pi}{\log(\Lambda)}\mathbb{Z},$$

  while $\mathrm{Mod}(\widehat{\mathbb{G}}) = \bigcap_{\Lambda \in \mathrm{Sp}(F^*F) \setminus \{\lambda^{-1}\}} \frac{2\pi}{\log \lambda + \log(\Lambda)}\mathbb{Z}$, where $\lambda = \sqrt{\frac{\mathrm{Tr}((F^*F)^{-1})}{\mathrm{Tr}(F^*F)}}$.
- If $\mathbb{G}$ is not of Kac type then $L^\infty(\mathbb{G})$ is a type III_μ factor for some $\mu \in]0,1]$ and $T^\sigma_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\sigma_{\mathrm{Inn}}(\mathbb{G}) = \frac{2\pi}{\log \mu}\mathbb{Z}$ (otherwise $T^\sigma_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\sigma_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$).

Some comments

- In our previous work for any subgroup Γ of $\mathbb{R}$ we constructed second countable compact quantum group $\mathbb{K}$ such that $T^{\tau}_{\mathrm{Inn}}(\mathbb{K}) = \Gamma$.
- The invariants were helpful in showing that for any $\lambda \in]0, 1]$ thee are uncountably many pairwise non-isomorphic compact quantum groups $\mathbb{G}$ with $L^{\infty}(\mathbb{G})$ isomorphic to the injective factor of type III_{λ}.
- The equality $T^{\tau}(\mathrm{U}_F^+) = T^{\tau}_{\mathrm{Inn}}(\mathrm{U}_F^+)$ says that the compact quantum group U_F^+ belongs to the class for which the following statement is true:

Conjecture $(*)$

If $\mathbb{G}$ is a second countable compact quantum group and $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$ then $\mathbb{G}$ is of Kac type.

- We were able to prove that this conjecture is true for many compact quantum groups including duals of second countable type I discrete quantum groups (e.g. q-deformations of compact semisimple Lie groups).

More Examples

Example: q-deformations

Let G be a compact semisimple Lie group with root system Φ and let $q \in]0,1[$.

- Since G_q is compact we again have

$$\operatorname{Mod}(G_q) = T^{\tau}_{\mathrm{Inn}}(\widehat{G_q}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\widehat{G_q}) = T^{\sigma}_{\mathrm{Inn}}(\widehat{G_q}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\widehat{G_q}) = \mathbb{R}.$$

- Furthermore $T^{\sigma}_{\mathrm{Inn}}(G_q) = T^{\sigma}_{\overline{\mathrm{Inn}}}(G_q) = \mathbb{R}$ because $\mathrm{C}(G_q)$ is a C*-algebra of type I.
- We have $T^{\tau}(G_q) = \frac{\pi}{\log q}\mathbb{Z}$ and

$$T^{\tau}_{\mathrm{Inn}}(G_q) = T^{\tau}_{\overline{\mathrm{Inn}}}(G_q) = \operatorname{Mod}(\widehat{G_q}) = \frac{\pi}{\Upsilon_{\Phi}\log q}\mathbb{Z},$$

  where Υ_{Φ} is a positive integer determined by Lie-theoretic data (see next two slides).

More examples

Example: q-deformations (continued)

- Let $\Phi = \Phi_1 \cup \cdots \cup \Phi_l$ be the decomposition of Φ into irreducible parts. Then

$$\Upsilon_\Phi = \gcd(\Upsilon_{\Phi_1}, \ldots, \Upsilon_{\Phi_l}).$$

- We have

type	**group**	**range of n**	Υ_Φ	$T^\tau_{\mathrm{Inn}}(\mathbb{G}_q)$
A_n	$\mathrm{SU}(n+1)$	$n \geqslant 1$ odd	1	$\frac{\pi}{\log q}\mathbb{Z}$
		$n \geqslant 1$ even	2	$\frac{\pi}{2\log q}\mathbb{Z}$
B_n	$\mathrm{Spin}(2n+1)$	$n \geqslant 2$ odd	1	$\frac{\pi}{\log q}\mathbb{Z}$
		$n \geqslant 2$ even	2	$\frac{\pi}{2\log q}\mathbb{Z}$
C_n	$\mathrm{Sp}(2n)$	$n \geqslant 3$	2	$\frac{\pi}{2\log q}\mathbb{Z}$
D_n	$\mathrm{Spin}(2n)$	$n \geqslant 4$ odd	2	$\frac{\pi}{2\log q}\mathbb{Z}$
		$n \geqslant 4$ even	1	$\frac{\pi}{\log q}\mathbb{Z}$

More examples

Example: q-deformations (continued)

- And for the exceptional cases we have
  - type E_6: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{2\log q}\mathbb{Z}$,
  - type E_7: $\Upsilon_\Phi = 1$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{\log q}\mathbb{Z}$,
  - type E_8: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{2\log q}\mathbb{Z}$,
  - type F_4: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{2\log q}\mathbb{Z}$,
  - type G_2: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{2\log q}\mathbb{Z}$.

Special case

- Consider the compact quantum group $\mathrm{SU}_q(3)$.
- Then $\Upsilon_\Phi = 2$, so
$$T^\tau_{\mathrm{Inn}}\big(\mathrm{SU}_q(3)\big) = \tfrac{\pi}{2\log q}\mathbb{Z},$$
while $T^\tau\big(\mathrm{SU}_q(3)\big) = \tfrac{\pi}{\log q}\mathbb{Z}$.
- This means that there are non-trivial inner scaling automorphisms.
- $\mathrm{SU}_q(3)$ does not have non-trivial one-dimensional representations, so these scaling automorphisms are not implemented by a group-like element.

Proposition

Let G be such that $\Upsilon_\Phi = 2$. Then a unitary implementing the scaling automorphism for $t = \frac{\pi}{2\log q}$ does not belong to $\mathrm{C}(G_q)$. In particular, the restriction of this automorphism to $\mathrm{C}(G_q)$ is not inner.

And now for something completely different

Proposition

Let Γ be a discrete group. Then the following are equivalent:

1. Γ is i.c.c.,
2. $L(\Gamma)$ is a factor,
3. $\Delta_{\hat{\Gamma}}^{(n)}\bigl(L(\Gamma)\bigr)' \cap \underbrace{L(\Gamma)\,\overline{\otimes}\cdots\overline{\otimes}\,L(\Gamma)}_{n+1} = \mathbb{C}\mathbb{1}$ for some $n \in \mathbb{N}$,
4. $\Delta_{\hat{\Gamma}}^{(n)}\bigl(L(\Gamma)\bigr)' \cap \underbrace{L(\Gamma)\,\overline{\otimes}\cdots\overline{\otimes}\,L(\Gamma)}_{n+1} = \mathbb{C}\mathbb{1}$ for all $n \in \mathbb{N}$.

I.c.c.-type conditions

Proposition

Let $\mathbb{G}$ be a locally compact quantum group and assume that

$$\Delta_{\mathbb{G}}^{(n)}\bigl(L^\infty(\mathbb{G})\bigr)' \cap \underbrace{L^\infty(\mathbb{G})\,\overline{\otimes}\,\cdots\,\overline{\otimes}\,L^\infty(\mathbb{G})}_{n+1} = \mathbb{C}\mathbb{1}$$

for some $n \in \mathbb{N}$. Then $L^\infty(\mathbb{G})$ is a factor.

Definition

Let $\mathbb{\Gamma}$ be a discrete quantum group. We say that $\mathbb{\Gamma}$ is n-i.c.c. if

$$\Delta_{\hat{\mathbb{\Gamma}}}^{(n)}\bigl(L^\infty(\hat{\mathbb{\Gamma}})\bigr)' \cap \underbrace{L^\infty(\hat{\mathbb{\Gamma}})\,\overline{\otimes}\,\cdots\,\overline{\otimes}\,L^\infty(\hat{\mathbb{\Gamma}})}_{n+1} = \mathbb{C}\mathbb{1}.$$

I.C.C.-TYPE CONDITIONS

PROPOSITION

Let $\mathbb{\Gamma}$ be a discrete quantum group. If $\mathbb{\Gamma}$ is n-i.c.c. for some n then $\mathbb{\Gamma}$ is m-i.c.c. for all natural $m \leqslant n$.

THEOREM

Let $\mathbb{G}$ be a second countable compact quantum group whose dual is 1-i.c.c. Then conjecture $(*)$ holds for $\mathbb{G}$.

Example

- Recall that $\operatorname{Irr} \mathrm{U}_F^+ = \mathbb{Z}_+ \star \mathbb{Z}_+$ with the two copies of $\mathbb{Z}_+$ generated by the class α of the defining representation and $\beta = \overline{\alpha}$.
- For $x \in \mathbb{Z}_+ \star \mathbb{Z}_+$ put

$$D_{x,n} = \begin{cases} \|\rho_x^2 - \mathbb{1}\| \frac{\|\rho_x\|^{2(n+1)} - 1}{\|\rho_x\|^2 - 1} & \rho_x \neq \mathbb{1} \\ 0 & \rho_x = \mathbb{1} \end{cases}.$$

- Let $D_n = \max\{D_{\alpha\beta,n}, D_{\beta\alpha,n}, D_{\alpha^2\beta,n}\}$.

Theorem

If $D_n < 1 - \frac{1}{\sqrt{2}}$ and $\frac{2(7-4D_n)D_n}{2(1-D_n)^2-1} < \frac{1}{\sqrt{n+1}}$ then $\widehat{\mathrm{U}_F^+}$ is n-i.c.c.

Example (continued)

Theorem

Take $n \in \mathbb{N}$ and write $c = \max\Bigl\{\bigl\|\lambda F^*F - \mathbb{1}\bigr\|, \bigl\|(\lambda F^*F)^{-1} - \mathbb{1}\bigr\|\Bigr\}$, where $\lambda = \sqrt{\frac{\mathrm{Tr}((F^*F)^{-1})}{\mathrm{Tr}(F^*F)}}$. If

$$\sqrt{n}(n+1)c(2+c)(1+c)^{4+6n} < \tfrac{1}{72}$$

then $\widehat{\mathrm{U}_F^+}$ is n-i.c.c.

Post-doc position in Warsaw

- A position for one year starting March 2024 will be announced tomorrow.
- Please e-mail me if you are interested.

Thank you for your attention