Homomorphisms of locally compact quantum groups, quantum subgroups and integrability

Great Plains Operator Theory Symposium
University of Illinois at Urbana-Champaign

Piotr M. Sołtan
(joint work with P. Kasprzak & F. Khosravi)

Department of Mathematical Methods in Physics
Faculty of Physics, University of Warsaw

May 25, 2016

Definition

Let

- M be a von Neumann algebra,
- $\Delta\colon \mathsf{M} \longrightarrow \mathsf{M}\,\bar{\otimes}\,\mathsf{M}$ a be normal unital $*$-homomorphism such that
$$(\Delta \otimes \mathrm{id})\circ\Delta = (\mathrm{id} \otimes \Delta)\circ\Delta,$$
- $\boldsymbol{h}$, $\boldsymbol{h}_{\mathrm{L}}$ be n.f.s. weights on M such that
$$(\boldsymbol{h} \otimes \mathrm{id})\circ\Delta = \boldsymbol{h}(\cdot)\mathbb{1} \quad \text{and} \quad (\mathrm{id} \otimes \boldsymbol{h}_{\mathrm{L}})\circ\Delta = \boldsymbol{h}_{\mathrm{L}}(\cdot)\mathbb{1}.$$

Then we say that M is the algebra of essentially bounded functions on a locally compact quantum group. We write

$$\mathsf{M} = L^{\infty}(\mathbb{G})$$

and call $\mathbb{G}$ a **locally compact quantum group**.

G ⇠ a locally compact group,

1. **Classical groups** ($\mathbb{G} = G$)
   - $L^\infty(\mathbb{G}) := L^\infty(G)$,
   - $\Delta\colon L^\infty(\mathbb{G}) \longrightarrow L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{G}) \cong L^\infty(G \times G)$

$$\Delta(f)(x,y) = f(xy), \qquad f \in L^\infty(\mathbb{G}),\ x,y \in G,$$

   - with h and h_{L} the right and left Haar measure on G

$$\boldsymbol{h}(f) = \int_G f\,dh, \quad \boldsymbol{h}_{\mathrm{L}}(f) = \int_G f\,dh_{\mathrm{L}}, \qquad f \in L^\infty(\mathbb{G}).$$

2. **Duals of classical groups** ($\mathbb{G} = \widehat{G}$)
   - $L^\infty(\mathbb{G}) = \mathrm{vN}(G)$ ⇠ right group von Neumann algebra of G,
   - for $x \in G$ we write ρ_x for the right shift by x on $L^2(G)$

$$\Delta(\rho_x) = \rho_x \otimes \rho_x, \qquad x \in G,$$

   - $\boldsymbol{h} = \boldsymbol{h}_{\mathrm{L}}$ ⇠ evaluation at $e \in G$.

- $\mathbb{G}$ ⇠ locally compact quantum group,
- $\Delta_{\mathbb{G}}\colon L^{\infty}(\mathbb{G}) \longrightarrow L^{\infty}(\mathbb{G}) \bar{\otimes} L^{\infty}(\mathbb{G})$,
- $\boldsymbol{h}, \boldsymbol{h}_{\mathrm{L}}$ ⇠ right and left Haar weights on $L^{\infty}(\mathbb{G})$,
- $\widehat{\mathbb{G}}$ ⇠ the dual of $\mathbb{G}$,
- $L^{\infty}(\mathbb{G}), L^{\infty}(\widehat{\mathbb{G}}) \subset \mathrm{B}(L^2(\mathbb{G}))$, ⇠ GNS Hilbert space for $\boldsymbol{h}$
- $\mathrm{W}^{\mathbb{G}} \in L^{\infty}(\widehat{\mathbb{G}}) \bar{\otimes} L^{\infty}(\mathbb{G})$ ⇠ the Kac-Takesaki operator
- Slicing:

$$L^{\infty}(\mathbb{G}) = \left\{(\omega \otimes \mathrm{id})\mathrm{W}^{\mathbb{G}} \,\middle|\, \omega \in \mathrm{B}(L^2(\mathbb{G}))_*\right\}^{\overline{}\sigma},$$
$$L^{\infty}(\widehat{\mathbb{G}}) = \left\{(\mathrm{id} \otimes \omega)\mathrm{W}^{\mathbb{G}} \,\middle|\, \omega \in \mathrm{B}(L^2(\mathbb{G}))_*\right\}^{\overline{}\sigma}.$$

- Implementation of coproducts:

$$\Delta_{\mathbb{G}}(x) = \mathrm{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{G}*}, \qquad x \in L^{\infty}(\mathbb{G}),$$
$$\Delta_{\widehat{\mathbb{G}}}(y) = \boldsymbol{\sigma}\big(\mathrm{W}^{\mathbb{G}*}(\mathbb{1} \otimes y)\mathrm{W}^{\mathbb{G}}\big), \qquad y \in L^{\infty}(\widehat{\mathbb{G}}),$$

where $\boldsymbol{\sigma}$ is the flip.

- $\mathbb{G}$ ⇠ a locally compact quantum group,
- N ⇠ a von Neumann algebra.
- A **right action** of $\mathbb{G}$ on N is a normal unital injective $*$-homomorphism $\alpha\colon \mathsf{N} \longrightarrow \mathsf{N} \bar{\otimes} L^{\infty}(\mathbb{G})$ such that

$$(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta_{\mathbb{G}}) \circ \alpha.$$

- A **left action** is $\beta\colon \mathsf{N} \to L^{\infty}(\mathbb{G}) \bar{\otimes} \mathsf{N}$ with $(\mathrm{id} \otimes \alpha) \circ \alpha = (\Delta_{\mathbb{G}} \otimes \mathrm{id}) \circ \alpha$.
- An action can be **implemented**: $\mathsf{N} \subset \mathrm{B}(\mathscr{H})$, there is a unitary $V \in \mathrm{B}(\mathscr{H}) \bar{\otimes} L^{\infty}(\mathbb{G})$ such that

$$\alpha(x) = V(x \otimes \mathbb{1})V^{*}, \qquad x \in \mathsf{N}$$

  (similarly for left actions).
- Any action admits a **canonical implementation** (work of S. Vaes).

- An action $\alpha\colon \mathsf{N} \longrightarrow \mathsf{N} \bar{\otimes} L^\infty(\mathbb{G})$ is **integrable** if the operator valued weight $(\mathrm{id} \otimes \boldsymbol{h}_{\mathrm{L}}) \circ \alpha$ is semifinite (for a left action β we ask that $(\boldsymbol{h} \otimes \mathrm{id}) \circ \beta$ be semifinite).

Example $\mathbb{G} = G$ ←-- locally compact group

- α ←-- right action of G on a von Neumann algebra N:

$$\alpha\colon G \ni t \longmapsto \alpha_t \in \mathrm{Aut}(\mathsf{N}).$$

(- $\alpha\colon \mathsf{N} \longrightarrow \mathsf{N} \bar{\otimes} L^\infty(\mathbb{G}) \cong L^\infty(G, \mathsf{N})$, $\alpha(x)(t) = \alpha_t(x)$.)

- α is integrable if

$$\int_G \alpha_t(x)\, dh_{\mathrm{L}}(t)$$

belongs to N_+ (not $\mathsf{N}_+^{\mathrm{ext}}$) for a dense set of $x \in \mathsf{N}_+$.

- Integrability can be characterized in terms of the canonical implementation.

- $\mathbb{H}$, $\mathbb{G}$ ⇠ locally compact quantum groups.

THEOREM

There is a bijection between sets of

1. ***bicharacters** from $\mathbb{H}$ to $\mathbb{G}$, i.e. unitaries $V \in L^\infty(\widehat{\mathbb{G}}) \bar{\otimes} L^\infty(\mathbb{H})$ such that*

$$(\Delta_{\widehat{\mathbb{G}}} \otimes \mathrm{id})V = V_{23}V_{13} \quad \textit{and} \quad (\mathrm{id} \otimes \Delta_{\mathbb{H}})V = V_{12}V_{13},$$

2. ***right quantum group homomorphisms** from $\mathbb{H}$ to $\mathbb{G}$ i.e. actions $\alpha\colon\ L^\infty(\mathbb{G}) \longrightarrow L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{H})$ such that*

$$(\Delta_{\mathbb{G}} \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \alpha) \circ \Delta_{\mathbb{G}}.$$

DEFINITION

A **homomorphism** $\pi\colon \mathbb{H} \longrightarrow \mathbb{G}$ is an element of either of the sets of 1 or 2.

- $\mathbb{G} = G$, $\mathbb{H} = H$ ⇠-- locally compact groups,
- $\pi\colon H \longrightarrow G$ ⇠-- continuous homomorphism.
- The action α associated to π is

$$\alpha\colon H \longrightarrow \mathrm{Aut}\bigl(L^\infty(G)\bigr)$$

with

$$\bigl(\alpha_h(f)\bigr)(g) = f\bigl(g\pi(h)\bigr), \qquad h \in H,\ g \in G,\ f \in L^\infty(G).$$

- The corresponding bicharacter is

$$V \in L^\infty(\widehat{\mathbb{G}}) \bar{\otimes} L^\infty(H) \cong L^\infty\bigl(H, \mathrm{vN}(G)\bigr)$$

defined by

$$V(h) = \rho_{\pi(h)}, \qquad h \in H.$$

- As before: $\mathbb{H}$, $\mathbb{G}$ ⇠ l.c.q.g.'s, $\pi\colon \mathbb{H} \to \mathbb{G}$ ⇠ homomorphism.
- With π we associate
  - a bicharacter $V \in L^\infty(\widehat{\mathbb{G}}) \bar{\otimes} L^\infty(\mathbb{H})$,
  - an action $\alpha\colon L^\infty(\mathbb{G}) \longrightarrow L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{H})$.
- Then V implements α:

$$\alpha(x) = V(x \otimes \mathbb{1})V^*, \qquad x \in L^\infty(\mathbb{G}).$$

- Moreover V is the canonical implementation of α.

DEFINITION

A homomorphism $\pi\colon \mathbb{H} \longrightarrow \mathbb{G}$ identifies $\mathbb{H}$ with a **closed quantum subgroup** of $\mathbb{G}$ if there is an

- injective,
- unital,
- normal

$*$-homomorphism $\gamma\colon\ L^{\infty}(\widehat{\mathbb{H}}) \longrightarrow L^{\infty}(\widehat{\mathbb{G}})$ such that

$$V = (\gamma \otimes \mathrm{id})\mathrm{W}^{\mathbb{H}}$$

($\mathrm{W}^{\mathbb{H}}$ is the Kac-Takesaki operator of $\mathbb{H}$).

- There is a (potentially) weaker C*-algebraic notion of a **Woronowicz-closed** quantum subgroup.
- A closed quantum subgroup is Woronowicz-closed.
- The two notions are equivalent in many cases, but there is no proof of this in the general case (so far).
- If $\mathbb{H}$ is a Woronowicz-closed quantum subgroup of $\mathbb{G}$ then the action α of $\mathbb{H}$ on $\mathbb{G}$ is **free**: the set

$$\left\{(\omega\otimes\mathrm{id})\alpha(x)\,\middle|\,x\in L^\infty(\mathbb{G}),\ \omega\in\mathrm{B}(L^2(\mathbb{G}))_*\right\}$$

generates the von Neumann algebra $L^\infty(\mathbb{H})$.

Definition

Let $\mathbb{H}$, $\mathbb{G}$ be locally compact quantum groups. We say that $\mathbb{H}$ is an **open quantum subgroup** of $\mathbb{G}$ if there is a surjective normal $*$-homomorphism $\theta\colon L^\infty(\mathbb{G}) \to L^\infty(\mathbb{H})$ such that

$$\Delta_{\mathbb{H}} \circ \theta = (\theta \otimes \theta) \circ \Delta_{\mathbb{G}}.$$

Theorem (Kalantar-Kasprzak-Skalski)

An open quantum subgroup $\mathbb{H}$ of $\mathbb{G}$ is closed. The corresponding bicharacter is

$$V = (\mathrm{id} \otimes \theta)\mathrm{W}^{\mathbb{G}} \in L^\infty(\widehat{\mathbb{G}}) \bar{\otimes} L^\infty(\mathbb{H}).$$

- $\pi\colon \mathbb{H} \longrightarrow \mathbb{G}$ ⇠ homomorphism,
- V ⇠ corresponding bicharacter.

PROPOSITION

There is a locally compact quantum group $\mathbb{K}$ such that

$$L^\infty(\mathbb{K}) = \left\{(\mathrm{id}\otimes\zeta)V \,\middle|\, \zeta\in \mathrm{B}(L^2(\mathbb{H}))_*\right\}^{\sigma-\text{—}}$$

and

$$\Delta_{\mathbb{K}} = \Delta_{\widehat{\mathbb{G}}}\big|_{L^\infty(\mathbb{K})}$$

DEFINITION

The quantum group $\overline{\mathrm{im}\,\pi}$ is the dual of the quantum group $\mathbb{K}$. In particular

$$L^\infty\big(\widehat{\overline{\mathrm{im}\,\pi}}\big) = \left\{(\mathrm{id}\otimes\zeta)V \,\middle|\, \zeta\in \mathrm{B}(L^2(\mathbb{H}))_*\right\}^{\sigma-\text{—}}.$$

- $\pi\colon \mathbb{H} \longrightarrow \mathbb{G}$, V ⇠ as before.

Proposition

There is a locally compact quantum group $\mathbb{L}$ such that

$$L^\infty(\mathbb{L}) = \left\{(\phi\otimes\mathrm{id})V \,\middle|\, \phi\in \mathrm{B}(L^2(\mathbb{G}))_*\right\}^{\overline{\sigma}}$$

and

$$\Delta_{\mathbb{L}} = \Delta_{\mathbb{H}}\big|_{L^\infty(\mathbb{L})}$$

Definition

We call $\mathbb{L}$ the quantum group $\mathbb{H}/\ker\pi$. In particular

$$L^\infty\big(\mathbb{H}/\ker\pi\big) = \left\{(\phi\otimes\mathrm{id})V \,\middle|\, \phi\in \mathrm{B}(L^2(\mathbb{G}))_*\right\}^{\overline{\sigma}}.$$

- $\pi\colon \mathbb{H} \longrightarrow \mathbb{G}$, V, α ⇠-- as before.
- We say that $\mathbb{H}/\ker\pi \cong \overline{\operatorname{im}\pi}$ if there is an isomorphism $\chi\colon\ L^\infty\big(\overline{\operatorname{im}\pi}\big) \longrightarrow L^\infty(\mathbb{H}/\ker\pi)$ such that

$$(\mathrm{id}\otimes\chi)(\mathrm{W}^{\overline{\operatorname{im}\pi}}) = V.$$

- Note:
  - $V \in L^\infty\big(\widehat{\overline{\operatorname{im}\pi}}\big)\,\bar{\otimes}\,L^\infty(\mathbb{H}/\ker\pi)$, ($V \in L^\infty(\widehat{\mathbb{G}})\,\bar{\otimes}\,L^\infty(\mathbb{H})$)
  - an isomorphism χ as above is necessarily unique.

THEOREM

The action α (corresponding to π) is integrable if and only if

1. $\widehat{\mathbb{H}/\ker}\,\pi$ *is an open subgroup of* $\widehat{\mathbb{H}}$,
2. $\mathbb{H}/\ker\pi \cong \overline{\operatorname{im}\pi}$.

- Point ① “means” that $\ker\pi$ is compact.

THEOREM (K. DE COMMER)

Let $\pi\colon \mathbb{H} \longrightarrow \mathbb{G}$ identify $\mathbb{H}$ with a closed quantum subgroup of $\mathbb{G}$. Then the associated action α is integrable.

COROLLARY

A Woronowicz-closed quantum subgroup $\mathbb{H}$ of $\mathbb{G}$ is a closed quantum subgroup if and only if the corresponding action

$$\alpha\colon\ L^\infty(\mathbb{G}) \longrightarrow L^\infty(\mathbb{G})\,\bar{\otimes}\ L^\infty(\mathbb{H})$$

is integrable.

PROOF.

$\Rightarrow$ Immediate from theorem.
$\Leftarrow$: $\mathbb{H}$ Woronowicz-closed implies that α is free, so $\mathbb{H}/\ker\pi = \mathbb{H}$. Integrability implies that $\mathbb{H}/\ker\pi \cong \overline{\operatorname{im}\pi}$ and $\overline{\operatorname{im}\pi}$ is closed. $\square$

- $\pi\colon H \longrightarrow G$ ⇠-- continuous homomorphism of l.c. groups.
- The associated action α is $H \ni h \longmapsto \alpha_h \in \mathrm{Aut}\bigl(L^\infty(G)\bigr)$:

$$\bigl(\alpha_h(f)\bigr)(g) = f\bigl(g\,\pi(h)\bigr), \qquad g \in G,\ f \in L^\infty(G).$$

- Integrability of α means that

$$G \ni g \longmapsto \int_H \bigl(\alpha_h(f)\bigr)(g)\,dh$$

is in $L^\infty(G)_+$ for a dense set of functions $f \in L^\infty(G)_+$.

COROLLARY

1. *α is integrable if and only if $\ker\pi$ is compact and $\mathrm{im}\,\pi$ is closed and topologically isomorphic to $H/\ker\pi$.*
2. *When π is injective, α is integrable if and only if the image of π is closed and π is a homeomorphism onto its image.*