Operators with the property $\operatorname{Sp} R = \operatorname{Sp} S = \operatorname{Sp}(R + S)$ and quantum exponential functions

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Report on work with W. Pusz & S.L. Woronowicz

Consider

- a closed subset $\Sigma \subset \mathbb{C}$,
- a pair of operators (R, S) on a Hilbert space such that

$$\left(\begin{array}{c} R \text{ and } S \text{ are } \underline{\text{normal}}, \\ \operatorname{Sp} R,\ \operatorname{Sp} S \subset \Sigma, \\ R \text{ and } S \text{ satisfy some}, \\ \text{commutation relations} \end{array} \right)$$

  and the relations imply that $S + R$ is a densely defined closable operator,
- denote by $S \dot{+} R$ the closure of $S + R$.

Questions:

- Is $S \dot{+} R$ normal?
- Is $\operatorname{Sp}(S \dot{+} R)$ contained in Σ?

These are questions about the relations satisfied by (R, S) and about Σ.

Obvious example

- R and S strongly commute,
- Σ is an additive subgroup of $\mathbb{C}$.

"Quantum" examples

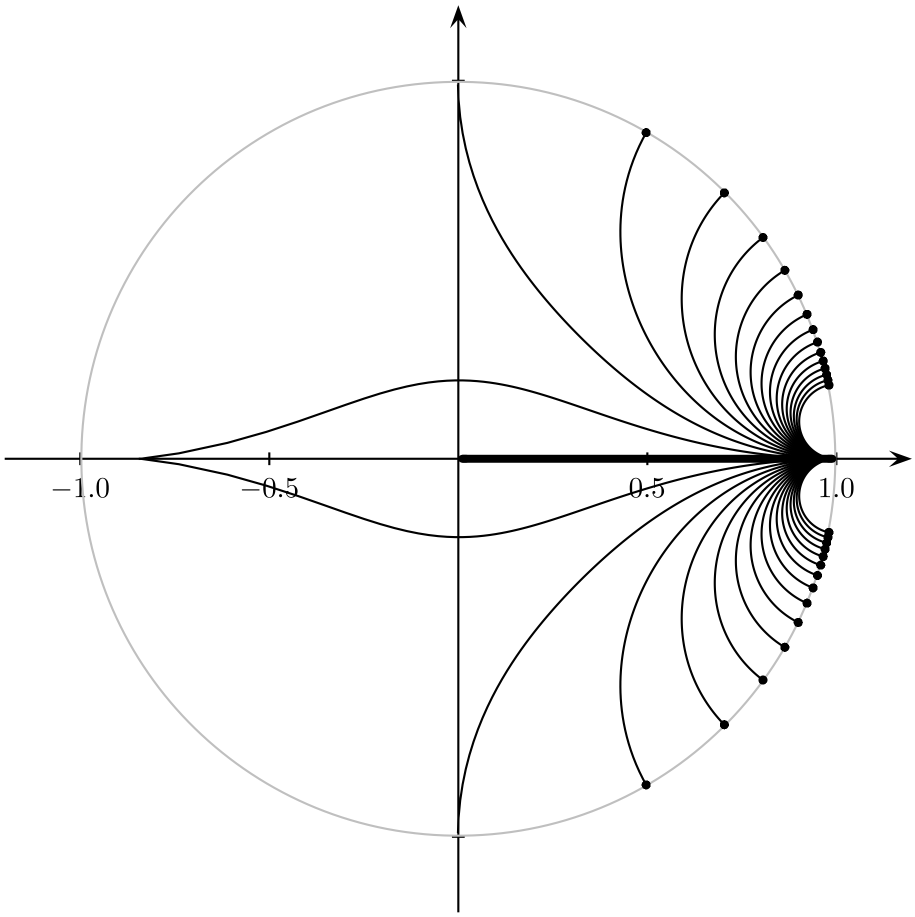
- Σ is the closure in $\mathbb{C}$ of a special <u>multiplicative</u> subgroup of $\mathbb{C} \setminus \{0\}$,
- R and S satisfy

$$SR = q^2 RS \qquad \text{and} \qquad SR^* = R^* S$$

for a <u>special</u> complex number $q \neq 1$.

Remark The proposed relations imply that R and S have to be unbounded (or zero) – cf. Fuglede-Putnam theorem.

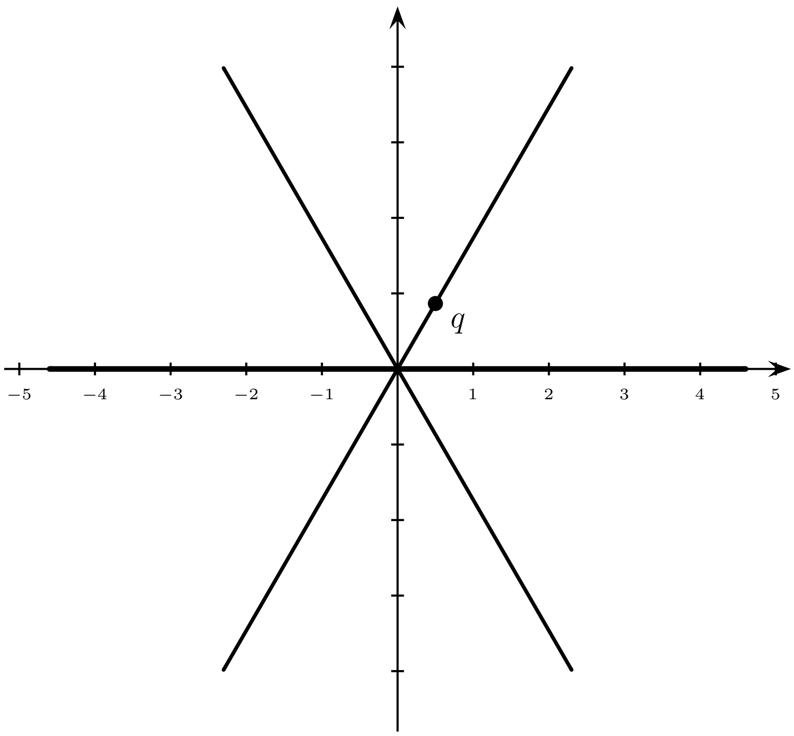
Chose q from the set



then

- Let Γ be the <u>multiplicative</u> subgroup of $\mathbb{C} \setminus \{0\}$ generated by q and $\left\{q^{it} : t \in \mathbb{R}\right\}$.
- Define
$$\Sigma = \overline{\Gamma} = \Gamma \cup \{0\}.$$

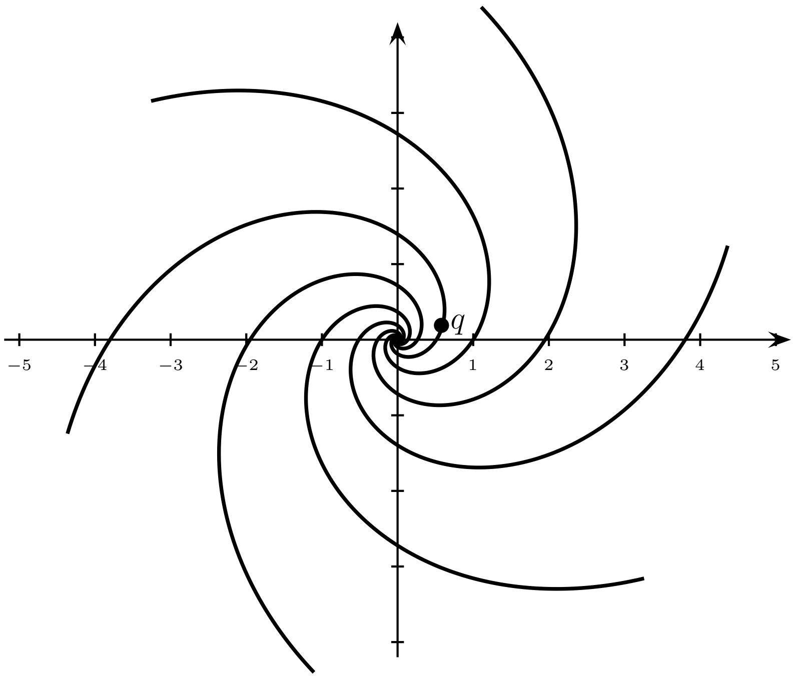
Then Σ looks like this



if we chose q as the root of unity

$$q = e^{\frac{2\pi i}{6}},$$

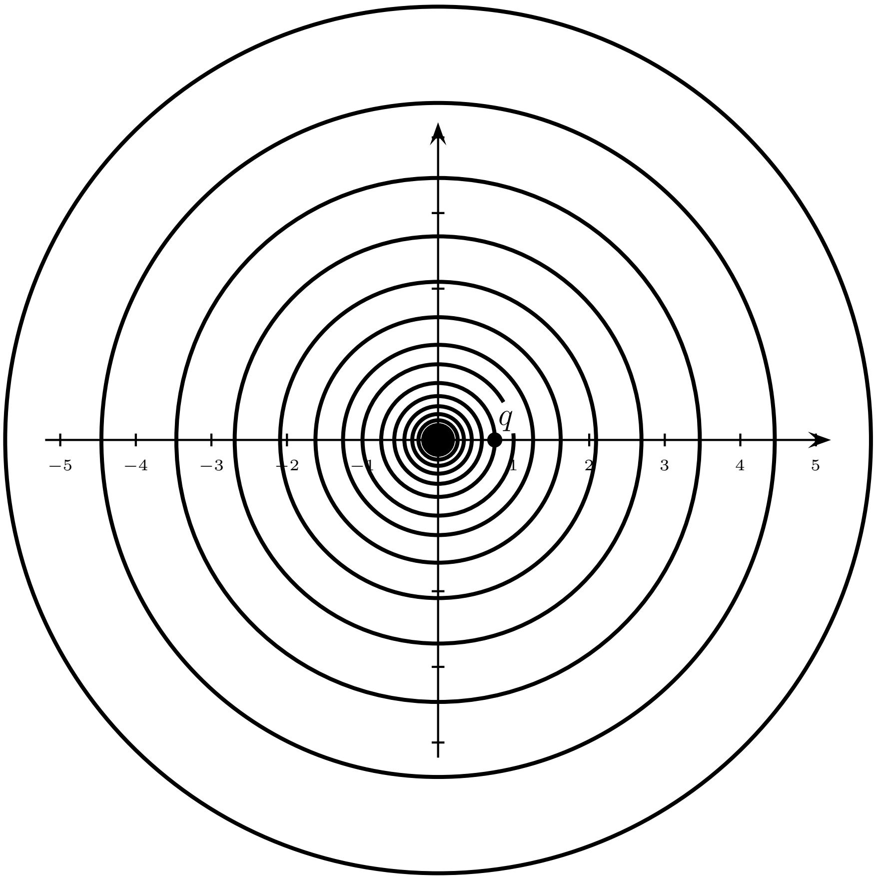
or like this



if we chose

$$q = e^{\left(-\frac{3}{2} - i\frac{6}{2\pi}\right)^{-1}},$$

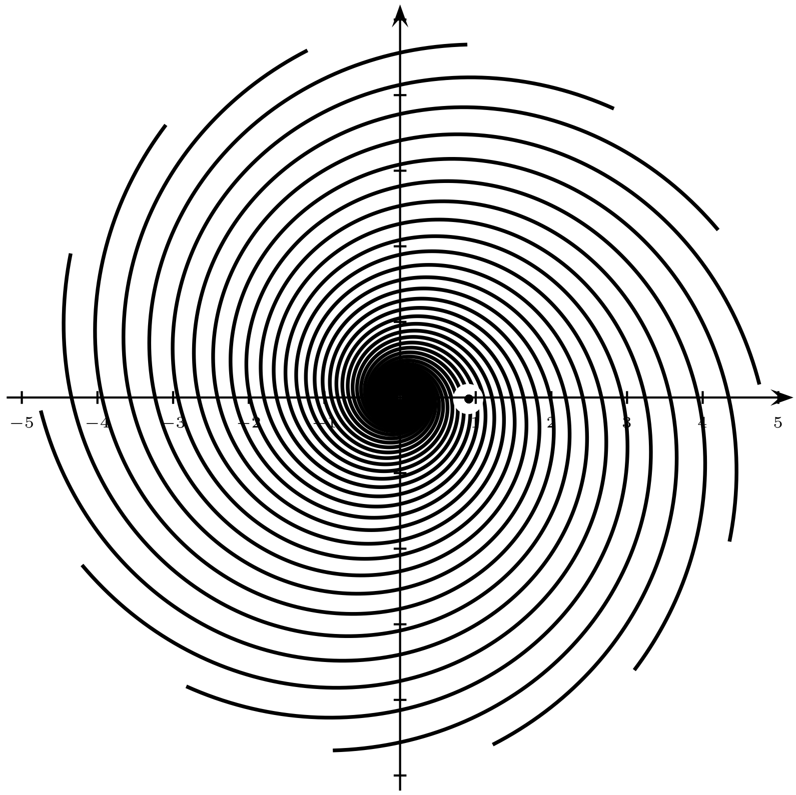
or like this



if we choose

$$q = e^{-\frac{1}{4}} \simeq 0.78.$$

Another example



Here

$$q = e^{\left(-10+i\frac{14}{2\pi}\right)^{-1}}.$$

The group Γ is very special.

- If q is real then $\Gamma \simeq \mathbb{Z} \times \boldsymbol{S}^1$.
- For other q we have $\Gamma \simeq \mathbb{Z}_N \times \mathbb{R}$, where N is even.

In both cases Γ is self dual and we have a symmetric nondegenerate bicharacter on Γ, i.e.

$$\chi \colon \Gamma \times \Gamma \longrightarrow \boldsymbol{S}^1$$

which establishes the self duality.

The commutation relations

- $\ker R = \ker S = \{0\}$,
- for any $\gamma, \gamma' \in \Gamma$ we have

  $$\boxed{\chi(S,\gamma)\chi(R,\gamma') = \chi(\gamma,\gamma')\chi(R,\gamma')\chi(S,\gamma)}$$

  This is called the Weyl relation (cf. Weyl form of CCRs).

  Remember R and S are normal and $\operatorname{Sp} R$, $\operatorname{Sp} S \subset \Sigma = \Gamma \cup \{0\}$.

Consequences of the relations

Putting different γ and γ' in the Weyl relation

$$\chi(S,\gamma)\chi(R,\gamma') = \chi(\gamma,\gamma')\chi(R,\gamma')\chi(S,\gamma)$$

one arrives at

$$\begin{aligned}
\mathrm{Phase}(S)|R| &= |q||R|\mathrm{Phase}(S),\\
|S|\mathrm{Phase}(R) &= |q|\mathrm{Phase}(R)|S|,\\
\mathrm{Phase}(S)\mathrm{Phase}(R) &= \mathrm{Phase}(q)\mathrm{Phase}(R)\mathrm{Phase}(S),\\
|S|^{it}|R|^{it'} &= \mathrm{Phase}(q)^{tt'}|R|^{it'}|S|^{it},
\end{aligned}$$

for all $t, t' \in \mathbb{R}$.

Caution: The above relations do not contain all the information. It does not follow from them that spectra of R and S are contained in Σ.

Theorem Let (R, S) be as before.

- Compositions $S \circ R$, $R \circ S$, $S \circ R^*$ and $R^* \circ S$ are closable and their closures satisfy

$$SR = q^2 RS, \qquad SR^* = R^* S.$$

- $S + R$ is a closable operator,
- There exists a continuous function

$$F_q \colon \Sigma \longrightarrow \boldsymbol{S}^1$$

such that

$$\begin{aligned} S \dot{+} R &= F_q(RS^{-1})^* S F_q(RS^{-1}) \\ &= F_q(R^{-1}S) R F_q(R^{-1}S)^*. \end{aligned}$$

In particular $S \dot{+} R$ is a normal operator with $\operatorname{Sp}(S \dot{+} R) \subset \Sigma$.

- F_q satisfies the <u>exponential equation</u>

$$F_q(S \dot{+} R) = F_q(R) F_q(S).$$

Remark F_q is essentially the only function with the exponential property.

Nice fact: the relations

$$\mathrm{Phase}(S)|R| = |q||R|\mathrm{Phase}(S),$$

$$|S|\mathrm{Phase}(R) = |q|\mathrm{Phase}(R)|S|,$$

$$\mathrm{Phase}(S)\mathrm{Phase}(R) = \mathrm{Phase}(q)\mathrm{Phase}(R)\mathrm{Phase}(S),$$

$$|S|^{it}|R|^{it'} = \mathrm{Phase}(q)^{tt'}|R|^{it'}|S|^{it},$$

supplemented by

- $S + R$ has a normal extension,
- $1 \in \mathrm{Sp}\, R$.

imply the Weyl relation.