# Invariants of quantum groups related to the scaling and modular groups

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# The setup

- Let $\mathbb{G}$ be a locally compact quantum group and left Haar measure $\varphi$.
- Let $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ be the modular group of $\varphi$ and let $\{\tau_t^\mathbb{G}\}_{t\in\mathbb{R}}$ denote the scaling group of $\mathbb{G}$.
- Let $\delta$ be the modular element of $\mathbb{G}$.
- Denote the group of inner automorphisms of the von Neumann algebra $L^\infty(\mathbb{G})$ by $\mathrm{Inn}(L^\infty(\mathbb{G}))$ and the group of approximately inner automorphisms of $L^\infty(\mathbb{G})$ by $\overline{\mathrm{Inn}}(L^\infty(\mathbb{G}))$.

# The invariants

**Definition**

We define

$$
\begin{aligned}
T^{\tau}(\mathbb{G}) &= \bigl\{t \in \mathbb{R} \bigm| \tau_t^{\mathbb{G}} = \mathrm{id}\bigr\},\\
T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) &= \bigl\{t \in \mathbb{R} \bigm| \tau_t^{\mathbb{G}} \in \mathrm{Inn}\bigl(L^{\infty}(\mathbb{G})\bigr)\bigr\},\\
T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) &= \bigl\{t \in \mathbb{R} \bigm| \tau_t^{\mathbb{G}} \in \overline{\mathrm{Inn}}\bigl(L^{\infty}(\mathbb{G})\bigr)\bigr\},\\
T^{\sigma}(\mathbb{G}) &= \bigl\{t \in \mathbb{R} \bigm| \sigma_t^{\varphi} = \mathrm{id}\bigr\},\\
T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) &= \bigl\{t \in \mathbb{R} \bigm| \sigma_t^{\varphi} \in \mathrm{Inn}\bigl(L^{\infty}(\mathbb{G})\bigr)\bigr\},\\
T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) &= \bigl\{t \in \mathbb{R} \bigm| \sigma_t^{\varphi} \in \overline{\mathrm{Inn}}\bigl(L^{\infty}(\mathbb{G})\bigr)\bigr\},\\
\mathrm{Mod}(\mathbb{G}) &= \bigl\{t \in \mathbb{R} \bigm| \delta^{it} = \mathbb{1}\bigr\}.
\end{aligned}
$$

# Some properties of the invariants

- The sets $T^{\circ}_{\bullet}(\mathbb{G})$ are subgroups of $\mathbb{R}$ and are isomorphism invariants of the quantum group $\mathbb{G}$.
- $T^{\tau}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}})$.
- $\boldsymbol{T}^{\bullet}(\mathbb{G})$, $T^{\bullet}_{\overline{\mathrm{Inn}}}(\mathbb{G})$, and $\mathrm{Mod}(\mathbb{G})$ are closed.
- We would obtain the same groups $T^{\sigma}(\mathbb{G})$, $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G})$, and $T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G})$ if we chose the right Haar measure instead of the left one.
- $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G})$ is equal to the Connes' invariant $T(L^{\infty}(\mathbb{G}))$. Consequently, $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G})$ depends only on the von Neumann algebra $L^{\infty}(\mathbb{G})$. It is also the case for $T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G})$.

# Some properties of the invariants

**Proposition**

For any locally compact quantum group $\mathbb{G}$ we have

$$\begin{aligned} T^{\sigma}(\mathbb{G}) &= T^{\tau}(\mathbb{G}) \cap \operatorname{Mod}(\hat{\mathbb{G}}), \\ T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) \cap \operatorname{Mod}(\hat{\mathbb{G}}) &= T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) \cap \operatorname{Mod}(\hat{\mathbb{G}}), \\ T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) \cap \operatorname{Mod}(\hat{\mathbb{G}}) &= T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) \cap \operatorname{Mod}(\hat{\mathbb{G}}), \\ \operatorname{Mod}(\mathbb{G}) \cap \operatorname{Mod}(\hat{\mathbb{G}}) &\subset \tfrac{1}{2} T^{\tau}(\mathbb{G}). \end{aligned}$$

- The first equality above together with $T^{\tau}(\mathbb{G}) = T^{\tau}(\hat{\mathbb{G}})$ reduces the list to 11 (invariants $T^{\sigma}(\mathbb{G})$, $T^{\sigma}(\hat{\mathbb{G}})$ and $T^{\tau}(\hat{\mathbb{G}})$ are determined by the remaining ones).
- If $\mathbb{G}$ is compact then $\operatorname{Mod}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\hat{\mathbb{G}}) = T^{\sigma}_{\mathrm{Inn}}(\hat{\mathbb{G}}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\hat{\mathbb{G}}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\hat{\mathbb{G}}) = \mathbb{R}$.
- If additionally $L^{\infty}(\mathbb{G})$ is semifinite then $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = \mathbb{R}$.

# Examples

### Example: the quantum $E(2)$ group

With $\mathbb{G} = \mathrm{E}_q(2)$ for some $q \in ]0,1[$ we have

$$T^{\tau}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^{\sigma}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}}) = T^{\sigma}(\widehat{\mathbb{G}}) = \mathrm{Mod}(\widehat{\mathbb{G}}) = \frac{\pi}{\log q}\mathbb{Z},$$

$$T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathrm{Mod}(\mathbb{G}) = \mathbb{R}.$$

## Examples

### Example: quantum "$az+b$" groups

Let $\mathbb{G}$ be the quantum "$az+b$" group for the deformation parameter $q$ in one of the three cases:

1. $q = \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}$ with $N = 6, 8, \dots,$
2. $q \in \,]0, 1[,$
3. $q = \mathrm{e}^{1/\rho}$ with $\operatorname{Re}\rho < 0$, $\operatorname{Im}\rho = \frac{N}{2\pi}$ with $N = \pm 2, \pm 4, \dots.$

Then

$$T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = T^{\tau}_{\underline{\mathrm{Inn}}}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\tau}_{\underline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) = T^{\sigma}_{\underline{\mathrm{Inn}}}(\mathbb{G}) = T^{\sigma}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\underline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathbb{R},$$

$$T^{\tau}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}}) = T^{\sigma}(\mathbb{G}) = T^{\sigma}(\widehat{\mathbb{G}}) = \operatorname{Mod}(\mathbb{G}) = \operatorname{Mod}(\widehat{\mathbb{G}}) = \begin{cases} \{0\} & \text{in cases } ① \text{ and } ③ \\ \frac{\pi}{\log q}\mathbb{Z} & \text{in case } ② \end{cases}.$$

# Examples

## Example: $\mathrm{U}_F^+$

Let $\mathbb{G}$ be the quantum group $\mathrm{U}_F^+$. Then $L^\infty(\mathbb{G})$ is a full factor, so $\mathrm{Inn}(L^\infty(\mathbb{G})) = \overline{\mathrm{Inn}}(L^\infty(\mathbb{G}))$ (Vaes).

- $\mathbb{G}$ is compact, so $\mathrm{Mod}(\mathbb{G}) = T^\tau_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^\tau_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^\sigma_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^\sigma_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathbb{R}$.
- If $\mathbb{G}$ is not of Kac type ($\lambda F^*F \neq \mathbb{1}$) then

$$T^\tau_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\tau_{\mathrm{Inn}}(\mathbb{G}) = T^\tau(\mathbb{G}) = \bigcap_{\Lambda \in \mathrm{Sp}(F^*F \otimes (F^*F)^{-1}) \setminus \{1\}} \frac{2\pi}{\log(\Lambda)}\mathbb{Z},$$

  while $\mathrm{Mod}(\widehat{\mathbb{G}}) = \bigcap_{\Lambda \in \mathrm{Sp}(F^*F) \setminus \{\lambda^{-1}\}} \frac{2\pi}{\log\lambda + \log(\Lambda)}\mathbb{Z}$, where $\lambda = \sqrt{\frac{\mathrm{Tr}((F^*F)^{-1})}{\mathrm{Tr}(F^*F)}}$.
- If $\mathbb{G}$ is not of Kac type then $L^\infty(\mathbb{G})$ is a type $\mathrm{III}_\mu$ factor for some $\mu \in ]0,1]$ and $T^\sigma_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\sigma_{\mathrm{Inn}}(\mathbb{G}) = \frac{2\pi}{\log\mu}\mathbb{Z}$ (otherwise $T^\sigma_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\sigma_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$).

# Some comments

- In our previous work for any subgroup $\Gamma$ of $\mathbb{R}$ we constructed second countable compact quantum group $\mathbb{K}$ such that $T^{\tau}_{\mathrm{Inn}}(\mathbb{K}) = \Gamma$.
- The invariants were helpful in showing that for any $\lambda \in ]0,1]$ thee are uncountably many pairwise non-isomorphic compact quantum groups $\mathbb{G}$ with $L^{\infty}(\mathbb{G})$ isomorphic to the injective factor of type $\mathrm{III}_{\lambda}$.
- The equality $T^{\tau}(\mathrm{U}^{+}_{F}) = T^{\tau}_{\mathrm{Inn}}(\mathrm{U}^{+}_{F})$ says that the compact quantum group $\mathrm{U}^{+}_{F}$ belongs to the class for which the following statement is true:

**Conjecture (*)**

If $\mathbb{G}$ is a second countable compact quantum group and $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$ then $\mathbb{G}$ is of Kac type.

- We were able to prove that this conjecture is true for many compact quantum groups including duals of second countable type I discrete quantum groups (e.g. $q$-deformations of compact semisimple Lie groups).

# More Examples

## Example: $q$-deformations

Let $G$ be a compact semisimple Lie group with root system $\Phi$ and let $q \in \,]0,1[$.

- Since $G_q$ is compact we again have

$$\operatorname{Mod}(G_q) = T^{\tau}_{\mathrm{Inn}}(\widehat{G_q}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\widehat{G_q}) = T^{\sigma}_{\mathrm{Inn}}(\widehat{G_q}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\widehat{G_q}) = \mathbb{R}.$$

- Furthermore $T^{\sigma}_{\mathrm{Inn}}(G_q) = T^{\sigma}_{\overline{\mathrm{Inn}}}(G_q) = \mathbb{R}$ because $\mathrm{C}(G_q)$ is a $\mathrm{C}^*$-algebra of type I.
- We have $T^{\tau}(G_q) = \frac{\pi}{\log q}\mathbb{Z}$ and

$$T^{\tau}_{\mathrm{Inn}}(G_q) = T^{\tau}_{\overline{\mathrm{Inn}}}(G_q) = \operatorname{Mod}(\widehat{G_q}) = \frac{\pi}{\Upsilon_{\Phi}\log q}\mathbb{Z},$$

  where $\Upsilon_{\Phi}$ is a positive integer determined by Lie-theoretic data (see next two slides).

# More examples

## Example: $q$-deformations (continued)

- Let $\Phi = \Phi_1 \cup \cdots \cup \Phi_l$ be the decomposition of $\Phi$ into irreducible parts. Then

$$\Upsilon_\Phi = \gcd(\Upsilon_{\Phi_1}, \ldots, \Upsilon_{\Phi_l}).$$

- We have

| **type** | **group** | **range of $n$** | $\Upsilon_\Phi$ | $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q)$ |
|---|---|---|---|---|
| $A_n$ | $\mathrm{SU}(n+1)$ | $n \geqslant 1$ odd | 1 | $\frac{\pi}{\log q}\mathbb{Z}$ |
| | | $n \geqslant 1$ even | 2 | $\frac{\pi}{2\log q}\mathbb{Z}$ |
| $B_n$ | $\mathrm{Spin}(2n+1)$ | $n \geqslant 2$ odd | 1 | $\frac{\pi}{\log q}\mathbb{Z}$ |
| | | $n \geqslant 2$ even | 2 | $\frac{\pi}{2\log q}\mathbb{Z}$ |
| $C_n$ | $\mathrm{Sp}(2n)$ | $n \geqslant 3$ | 2 | $\frac{\pi}{2\log q}\mathbb{Z}$ |
| $D_n$ | $\mathrm{Spin}(2n)$ | $n \geqslant 4$ odd | 2 | $\frac{\pi}{2\log q}\mathbb{Z}$ |
| | | $n \geqslant 4$ even | 1 | $\frac{\pi}{\log q}\mathbb{Z}$ |

## More examples

### Example: $q$-deformations (continued)

- And for the exceptional cases we have
  - type $E_6$: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{2\log q}\mathbb{Z}$,
  - type $E_7$: $\Upsilon_\Phi = 1$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{\log q}\mathbb{Z}$,
  - type $E_8$: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{2\log q}\mathbb{Z}$,
  - type $F_4$: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{2\log q}\mathbb{Z}$,
  - type $G_2$: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{2\log q}\mathbb{Z}$.

## Special case

- Consider the compact quantum group $\mathrm{SU}_q(3)$.
- Then $\Upsilon_\Phi = 2$, so
$$T^\tau_{\mathrm{Inn}}\bigl(\mathrm{SU}_q(3)\bigr) = \tfrac{\pi}{2\log q}\mathbb{Z},$$
while $T^\tau\bigl(\mathrm{SU}_q(3)\bigr) = \tfrac{\pi}{\log q}\mathbb{Z}$.
- This means that there are non-trivial inner scaling automorphisms.
- $\mathrm{SU}_q(3)$ does not have non-trivial one-dimensional representations, so these scaling automorphisms are not implemented by a group-like element.

**Proposition**

Let $G$ be such that $\Upsilon_\Phi = 2$. Then a unitary implementing the scaling automorphism for $t = \tfrac{\pi}{2\log q}$ does not belong to $\mathrm{C}(G_q)$. In particular, the restriction of this automorphism to $\mathrm{C}(G_q)$ is not inner.

# And now for something completely different

**Proposition**

Let $\Gamma$ be a discrete group. Then the following are equivalent:

1. $\Gamma$ is i.c.c.,
2. $L(\Gamma)$ is a factor,
3. $\Delta_{\hat{\Gamma}}^{(n)}\bigl(L(\Gamma)\bigr)' \cap \underbrace{L(\Gamma)\,\overline{\otimes}\cdots\overline{\otimes}\,L(\Gamma)}_{n+1} = \mathbb{C}\mathbb{1}$ for some $n \in \mathbb{N}$,
4. $\Delta_{\hat{\Gamma}}^{(n)}\bigl(L(\Gamma)\bigr)' \cap \underbrace{L(\Gamma)\,\overline{\otimes}\cdots\overline{\otimes}\,L(\Gamma)}_{n+1} = \mathbb{C}\mathbb{1}$ for all $n \in \mathbb{N}$.

# I.c.c.-type conditions

## Proposition

Let $\mathbb{G}$ be a locally compact quantum group and assume that

$$\Delta_{\mathbb{G}}^{(n)}\bigl(L^\infty(\mathbb{G})\bigr)' \cap \underbrace{L^\infty(\mathbb{G})\,\bar{\otimes}\cdots\bar{\otimes}\,L^\infty(\mathbb{G})}_{n+1} = \mathbb{C}\mathbb{1}$$

for some $n \in \mathbb{N}$. Then $L^\infty(\mathbb{G})$ is a factor.

## Definition

Let $\mathbb{\Gamma}$ be a discrete quantum group. We say that $\mathbb{\Gamma}$ is $n$-i.c.c. if

$$\Delta_{\hat{\mathbb{\Gamma}}}^{(n)}\bigl(L^\infty(\hat{\mathbb{\Gamma}})\bigr)' \cap \underbrace{L^\infty(\hat{\mathbb{\Gamma}})\,\bar{\otimes}\cdots\bar{\otimes}\,L^\infty(\hat{\mathbb{\Gamma}})}_{n+1} = \mathbb{C}\mathbb{1}.$$

# I.c.c.-type conditions

**Proposition**

Let $\mathbb{\Gamma}$ be a discrete quantum group. If $\mathbb{\Gamma}$ is $n$-i.c.c. for some $n$ then $\mathbb{\Gamma}$ is $m$-i.c.c. for all natural $m \leqslant n$.

**Theorem**

Let $\mathbb{G}$ be a second countable compact quantum group whose dual is 1-i.c.c. Then conjecture $(*)$ holds for $\mathbb{G}$.

**Theorem**

Let $\mathbb{G}$ be a second countable compact quantum group whose dual is 1-i.c.c. and such that $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$. Then $\mathbb{G}$ is of Kac type.

- We have $\tau^{\mathbb{G}}_t = \mathrm{Ad}(b^{\mathrm{i}t})$ for some positive self-adjoint operator $b$.
- Furthermore, for any $x \in L^\infty(\mathbb{G})$ and any $t \in \mathbb{R}$

$$(b^{-\mathrm{i}t} \otimes b^{-\mathrm{i}t})\Delta_{\mathbb{G}}(b^{\mathrm{i}t})\Delta_{\mathbb{G}}(x)\Delta_{\mathbb{G}}(b^{-\mathrm{i}t})(b^{\mathrm{i}t} \otimes b^{\mathrm{i}t}) = (\tau^{\mathbb{G}}_{-t} \otimes \tau^{\mathbb{G}}_{-t})\Delta_{\mathbb{G}}\big(\tau^{\mathbb{G}}_t(x)\big) = \Delta(x),$$

  so $(b^{-\mathrm{i}t} \otimes b^{-\mathrm{i}t})\Delta(b^{\mathrm{i}t}) \in \Delta_{\mathbb{G}}(L^\infty(\mathbb{G}))' \cap L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} L^\infty(\mathbb{G}) = \mathbb{C}\mathbb{1}$.

- Thus $(b^{-\mathrm{i}t} \otimes b^{-\mathrm{i}t})\Delta(b^{\mathrm{i}t}) = z_t\mathbb{1}$ for some scalars $z_t$. Moreover $t \mapsto z_t$ is a continuous homomorphism, so $z_t = \lambda^{\mathrm{i}t}$ for some $\lambda > 0$.
- Put $B = \lambda b$. Then still $\tau^{\mathbb{G}}_t = \mathrm{Ad}(B^{\mathrm{i}t})$ and, additionally, $\Delta_{\mathbb{G}}(B^{\mathrm{i}t}) = B^{\mathrm{i}t} \otimes B^{\mathrm{i}t}$ for all $t$.

**Theorem**

Let $\mathbb{G}$ be a second countable compact quantum group whose dual is 1-i.c.c. and such that $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$. Then $\mathbb{G}$ is of Kac type.

- Next we calculate

$$(\boldsymbol{h}_{\mathbb{G}} \otimes \mathrm{id})\Delta_{\mathbb{G}}\left(\int\limits_{t-\frac{1}{n}}^{t+\frac{1}{n}} B^{\mathrm{i}s}\,\mathrm{d}s\right) = (\boldsymbol{h}_{\mathbb{G}} \otimes \mathrm{id})\int\limits_{t-\frac{1}{n}}^{t+\frac{1}{n}} (B^{\mathrm{i}s} \otimes B^{\mathrm{i}s})\,\mathrm{d}s = \int\limits_{t-\frac{1}{n}}^{t+\frac{1}{n}} \boldsymbol{h}_{\mathbb{G}}(B^{\mathrm{i}s})B^{\mathrm{i}s}\,\mathrm{d}s$$

- Multiplying by $2n$ and taking $\lim\limits_{n\to\infty}$ we obtain $\boldsymbol{h}_{\mathbb{G}}(B^{\mathrm{i}t})\mathbb{1} = \boldsymbol{h}_{\mathbb{G}}(B^{\mathrm{i}t})B^{\mathrm{i}t}$, so $B = \mathbb{1}$.
- It follows that $\tau^{\mathbb{G}}_t = \mathrm{id}$ for all $t$.

# Example

- Recall that $\operatorname{Irr}\mathrm{U}_F^+ = \mathbb{Z}_+ \star \mathbb{Z}_+$ with the two copies of $\mathbb{Z}_+$ generated by the class $\alpha$ of the defining representation and $\beta = \overline{\alpha}$.
- For $x \in \mathbb{Z}_+ \star \mathbb{Z}_+$ put

$$D_{x,n} = \begin{cases} \|\rho_x^2 - \mathbb{1}\| \frac{\|\rho_x\|^{2(n+1)}-1}{\|\rho_x\|^2-1} & \rho_x \neq \mathbb{1} \\ 0 & \rho_x = \mathbb{1} \end{cases}.$$

- Let $D_n = \max\bigl\{D_{\alpha\beta,n}, D_{\beta\alpha,n}, D_{\alpha^2\beta,n}\bigr\}$.

**Theorem**

If $D_n < 1 - \frac{1}{\sqrt{2}}$ and $\frac{2(7-4D_n)D_n}{2(1-D_n)^2-1} < \frac{1}{\sqrt{n+1}}$ then $\widehat{\mathrm{U}_F^+}$ is $n$-i.c.c.

## Example (continued)

**Theorem**

Take $n \in \mathbb{N}$ and write $c = \max\left\{\left\|\lambda F^*F - \mathbb{1}\right\|, \left\|(\lambda F^*F)^{-1} - \mathbb{1}\right\|\right\}$, where $\lambda = \sqrt{\frac{\operatorname{Tr}((F^*F)^{-1})}{\operatorname{Tr}(F^*F)}}$. If

$$\sqrt{n}(n+1)c(2+c)(1+c)^{4+6n} < \tfrac{1}{72}$$

then $\widehat{\mathrm{U}_F^+}$ is $n$-i.c.c.

Thank you for your attention