Some quantum spaces which are not quantum groups

Graph Algebras

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Quantum spaces and quantum groups

- A **compact quantum space** is an object of the category dual to the category of C*-algebras corresponding to a unital C*-algebra.
- We will denote such objects by symbols such as $\mathbb{X}$ or $\mathbb{G}$ and the corresponding C*-algebras by $\mathrm{C}(\mathbb{X})$, $\mathrm{C}(\mathbb{G})$.

Definitions

A **compact quantum group** is a quantum space $\mathbb{G}$ together with a unital $*$-homomorphism $\Delta\colon \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$ such that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

and the sets $\{\Delta(a)(\mathbb{1} \otimes b) \,|\, a, b \in \mathrm{C}(\mathbb{G})\}$ and $\{(a \otimes \mathbb{1})\Delta(b) \,|\, a, b \in \mathrm{C}(\mathbb{G})\}$ are linearly dense in $\mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$.

Example: $\mathrm{SU}_q(2)$

Fix $q \in [-1, 1]\setminus\{0\}$. We define the quantum space $\mathrm{SU}_q(2)$ by setting $\mathrm{C}(\mathrm{SU}_q(2))$ to the the universal C*-algebra generated by elements α and γ such that

$$\begin{bmatrix}\alpha & -q\gamma^* \\ \gamma & \alpha^*\end{bmatrix}$$

is unitary. The comultiplication $\Delta\colon \mathrm{C}(\mathrm{SU}_q(2)) \to \mathrm{C}(\mathrm{SU}_q(2)) \otimes \mathrm{C}(\mathrm{SU}_q(2))$ is defined by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Non-example: $\mathbb{T}^2_\theta$

The quantum torus $\mathbb{T}^2_\theta$ is defined by letting $\mathrm{C}(\mathbb{T}^2_\theta)$ be the universal C*-algebra generated by unitaries u and v such that $uv = \mathrm{e}^{2\pi\mathrm{i}\theta}vu$. There is no compact quantum group structure on $\mathbb{T}^2_\theta$.

Some additional structure of compact quantum groups

- Let $\mathbb{G}$ be a compact quantum group.
- By a theorem of Woronowicz there exists a unique state $\boldsymbol{h}$ on $\mathrm{C}(\mathbb{G})$ such that $(\mathrm{id}\otimes\boldsymbol{h})\Delta(a)=(\boldsymbol{h}\otimes\mathrm{id})\Delta(a)=\boldsymbol{h}(a)\mathbb{1}$ for all $a\in\mathrm{C}(\mathbb{G})$. This state is the **Haar measure** of $\mathbb{G}$. We say that $\mathbb{G}$ is of **Kac type** if $\boldsymbol{h}$ is a trace.
- The modular group of $\boldsymbol{h}$ is given by

$$\sigma_t^{\boldsymbol{h}}(a)=(f_{\mathrm{i}t}\otimes\mathrm{id}\otimes f_{\mathrm{i}t})\Delta(a)$$

  where $\{f_{\mathrm{i}t}\}_{t\in\mathbb{R}}$ is a certain family of characters on $\mathrm{C}(\mathbb{G})$.
- There exists another one parameter group $\{\tau_t\}_{t\in\mathbb{R}}$ of automorphisms of $\mathrm{C}(\mathbb{G})$ such that

$$\tau_t(a)=(f_{\mathrm{i}t}\otimes\mathrm{id}\otimes f_{-\mathrm{i}t})\Delta(a)$$

  for all a. $\{\tau_t\}_{t\in\mathbb{R}}$ is the **scaling group** of $\mathbb{G}$.
- $\mathbb{G}$ is of Kac type iff the its scaling group is trivial.

Unitary representations

- A unitary representation of $\mathbb{G}$ is a unitary matrix

$$U=\begin{bmatrix}U_{1,1}&\cdots&U_{1,n}\\\vdots&\ddots&\vdots\\U_{n,1}&\cdots&U_{n,n}\end{bmatrix}\in\mathsf{Mat}_{\mathrm{n}}\bigl(\mathrm{C}(\mathbb{G})\bigr)=\mathsf{Mat}_{\mathrm{n}}(\mathbb{C})\otimes\mathrm{C}(\mathbb{G})$$

  such that $\Delta(U_{i,j})=\sum\limits_{k=1}^{n}U_{i,k}\otimes U_{k,j}$.
- For example, if $\mathbb{G}=\mathrm{SU}_q(2)$ then

$$\begin{bmatrix}\alpha&-q\gamma^*\\\gamma&\alpha^*\end{bmatrix}$$

  is a unitary representation of $\mathbb{G}$.
- The $U_{i,j}$ are called **matrix elements** of the representation.

Unitary representations

- A representation $U \in \mathsf{Mat}_n(\mathbb{C}) \otimes \mathrm{C}(\mathbb{G})$ is **irreducible** if the only projections $P \in \mathsf{Mat}_n(\mathbb{C})$ satisfying $(P \otimes \mathbb{1})U = U(P \otimes \mathbb{1})$ are 0 and $\mathbb{1}$.
- Two representations U and V of $\mathbb{G}$ are **equivalent** if $(u \otimes \mathbb{1})U = V(u \otimes \mathbb{1})$ for some unitary matrix u.
- We let $\mathrm{Irr}\,\mathbb{G}$ denote the set of equivalence classes of irreps of $\mathbb{G}$ and for any $\alpha \in \mathrm{Irr}\,\mathbb{G}$ we fix $U^\alpha \in \alpha$ such that there exists a diagonal matrix $\rho_\alpha = \mathrm{diag}(\rho_{\alpha,1}, \ldots, \rho_{\alpha,n_\alpha})$ satisfying

$$\boldsymbol{h}\left(U^\alpha_{k,l} U^{\alpha}_{i,j}{}^*\right) = \frac{\delta_{k,i}\delta_{l,j}\rho_{\alpha,j}}{\mathrm{Tr}\,\rho_\alpha}$$

  where by n_α we denoted the **dimension** of U^α (the size of the matrix).
- The span of matrix elements of irreducible representations is dense in $\mathrm{C}(\mathbb{G})$.

The quantum disk

- The **quantum disk** $\mathbb{D}$ was defined by Sheu in 1991 (and studied in depth by Klimek & Leśniewski in 1992) by declaring that $\mathrm{C}(\mathbb{D})$ is the universal C*-algebra generated by z such that $z^*z - zz^* = \mu(\mathbb{1} - zz^*)(\mathbb{1} - z^*z)$ with μ a parameter in $]0, 1[$.
- The algebra turns out to be independent of μ and isomorphic to the Toeplitz algebra $\mathscr{T}$ which we will treat as represented on $\boldsymbol{\mathcal{H}} = \ell_2$ in the usual way.
- Furthermore, $\mathrm{C}(\mathbb{D}) = \mathscr{T}$ is a graph algebra (of the graph with two vertices and incidence matrix $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$) and fits into the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathscr{T} \xrightarrow{\ s\ } \mathrm{C}(\mathbb{T}) \longrightarrow 0.$$

- Assume there is a compact quantum group $\mathbb{G}$ with an isomorphism $\pi\colon \mathrm{C}(\mathbb{G}) \to \mathscr{T}$.
- Goal: arrive at a contradiction.

First observations

Proposition

The family $\{f_{it}\}_{t\in\mathbb{R}}$ of Woronowicz characters of $\mathbb{G}$ coincides with the composition of $\mathrm{C}(\mathbb{G})\xrightarrow{\pi}\mathscr{T}\xrightarrow{s}\mathrm{C}(\mathbb{T})$ and point evaluations. In particular, $\mathbb{G}$ is not of Kac type.

Proposition

For any $\alpha\in\operatorname{Irr}\mathbb{G}$ the image $\pi(U^{\alpha}_{i,j})\in\mathrm{B}(\boldsymbol{\mathcal{H}})$ of $U^{\alpha}_{i,j}$ is

- compact if $i\neq j$,
- Fredholm if $i=j$.

Two important operators

- Let $\omega = \boldsymbol{h} \circ \pi^{-1}\big|_{\mathcal{K}}$.
- Then there exists a positive trace class operator A on $\mathcal{H}$ such that $\omega(\,\cdot\,) = \mathrm{Tr}(\,\cdot\,A)$ and $\ker A = \{0\}$.

Fact

For any $x \in \mathscr{T}$ and $t \in \mathbb{R}$ we have $(\pi \circ \sigma_t^{\boldsymbol{h}} \circ \pi^{-1})(x) = A^{\mathrm{i}t} x A^{-\mathrm{i}t}$.

Theorem

There exists a strictly positive self-adjoint operator B on $\mathcal{H}$ such that $(\pi \circ \tau_t \circ \pi^{-1})(x) = B^{\mathrm{i}t} x B^{-\mathrm{i}t}$ for all $x \in \mathscr{T}$ and $t \in \mathbb{R}$. Moreover A and B strongly commute.

Notation for eigenspaces and eigenprojections

- Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$.
- We will denote the eigenspace $\ker(t\mathbb{1} - T)$ by the symbol $\mathcal{H}(T = t)$.
- The corresponding spectral projection will be $\chi(T = t)$.
- Note that if $\alpha \mapsto \chi(\alpha)$ is the logical evaluation function (returning 0 if the proposition α is false and 1 if it is true) then $\chi(T = t)$ is the application to T of the function
$$\mathbb{R} \ni s \longmapsto \chi(s = t) \in \{0, 1\}.$$
- The notation $\mathcal{H}(T = t)$ is modeled on the typical expressions known from probability theory.

- Since A is trace-class, we have
$$\mathcal{H} = \bigoplus_{q \in \mathrm{Sp}(A)} \mathcal{H}(A = q).$$
with $\dim \mathcal{H}(A = q) < \infty$ for every q.
- Since B commutes with A, it preserves all eigenspaces of A and hence we have
$$B = \bigoplus_{q \in \mathrm{Sp}(A)} \chi(A = q) B \chi(A = q).$$
- For $q \in \mathrm{Sp}(A)$ we write B_q for $\chi(A = q) B \chi(A = q)$.

Proposition

Fix $\alpha \in \operatorname{Irr} \mathbb{G}$ and $i \in \{1, \ldots, n_\alpha\}$. Then

1. $\pi(U^{\alpha}_{i,i}) \mathcal{H}(A = q) \subset \mathcal{H}(A = q\rho^2_{\alpha,i})$,
2. For any $\lambda \in \mathrm{Sp}(B_q)$ we have $\pi(U^{\alpha}_{i,i}) \mathcal{H}(B_q = \lambda) \subset \mathcal{H}\big(B_{q\rho^2_{\alpha,i}} = \lambda\big)$.

Theorem

The set $\bigcup\limits_{q\in\mathrm{Sp}(A)} \mathrm{Sp}(B_q)$ *is finite.*

- Statement ① allows us to conclude that $\ker\pi(U^{\alpha}_{i,i})$ splits into subspaces

$$\ker\pi(U^{\alpha}_{i,i}) = \bigoplus_{q\in\mathrm{Sp}(A)} \ker\pi(U^{\alpha}_{i,i}) \cap \mathcal{H}(A=q).$$

- Since $\pi(U^{\alpha}_{i,i})$ is Fredholm, only finitely many of those can be non-zero, so there exists $q_0\in\mathrm{Sp}(A)$ such that $\pi(U^{\alpha}_{i,i})$ is injective on $\mathcal{H}(A=q)$ for all $q<q_0$.
- Choose α and i so that $\rho_{\alpha,i}>1$ (this is $\mathbb{G}$ not being Kac type).
- Assume that λ belongs to $\bigcup\limits_{\substack{q\in\mathrm{Sp}(A)\\ q<q_0}} \mathrm{Sp}(B_q)$, so that $\lambda\in\mathrm{Sp}(B_{\tilde{q}})$ for some $\tilde{q}\in\mathrm{Sp}(A)$, $\tilde{q}<q_0$.
- Then, by ②, for any $\xi\in\mathcal{H}(B_{\tilde{q}}=\lambda)$ the vector $\pi(U^{\alpha}_{i,i})^n\xi$ belongs to $\mathcal{H}\big(B_{\tilde{q}\rho^{2n}_{\alpha,i}}=\lambda\big)$.

Theorem

The set $\bigcup_{q\in\mathrm{Sp}(A)}\mathrm{Sp}(B_q)$ *is finite.*

- Take n such that $\tilde{q}\rho_{\alpha,i}^{2(n-1)} < q_0$ and $\tilde{q}\rho_{\alpha,i}^{2n} \geqslant q_0$. Then

$$\pi(U^{\alpha}_{i,i})\pi(U^{\alpha}_{i,i})^{n-1}\xi \in \mathcal{H}\bigl(B_{\tilde{q}\rho^{2n}_{\alpha,i}} = \lambda\bigr) = \{0\}$$

  (because λ is not an eigenvalue of B_q for $q \geqslant q_0$).
- This contradicts injectivity of $\pi(U^{\alpha}_{i,i})$ on $\mathcal{H}\bigl(B_{\tilde{q}\rho^{2(n-1)}_{\alpha,i}} = \lambda\bigr)$.
- It follows that B is bounded.
- By some arcane results of the theory of compact quantum groups the scaling group must be trivial, i.e. $\mathbb{G}$ is of Kac type — a contradiction!

Remarks

1. Similar techniques yield the following

Theorem (Alexandru Chirvasitu, Jacek Krajczok & P.M.S.)

Let $\mathbb{G}$ *be a compact quantum group such that the* C^**-algebra* $\mathrm{C}(\mathbb{G})$ *fits into the exact sequence*

$$0 \longrightarrow \bigoplus_{i=1}^{N} \mathcal{K}(\mathcal{H}_i) \longrightarrow \mathrm{C}(\mathbb{G}) \longrightarrow \mathrm{C}(X) \longrightarrow 0$$

with X *a compact space. The* $\mathbb{G}$ *is finite* (*i.e.* $\dim \mathrm{C}(\mathbb{G}) < +\infty$).

2. It follows that the Podleś spheres do not carry a compact quantum group structure (for the quotient sphere $\mathrm{SU}_q(2)/\mathbb{T}$ this had been known before).

Remarks

3. For a compact quantum group $\mathbb{G}$ we let $\mathsf{L}^\infty(\mathbb{G})$ denote the strong closure of the image of $\mathrm{C}(\mathbb{G})$ under the GNS representation for $\boldsymbol{h}$. In this framework we have

Theorem (Jacek Krajczok & P.M.S.)

1. *There does not exist a non-trivial compact quantum group* $\mathbb{G}$ *such that* $\mathsf{L}^\infty(\mathbb{G})$ *is a type one factor.*
2. *There does not exist a compact quantum groups such that* $\mathsf{L}^\infty(\mathbb{G})$ *has a direct summand wich is a type* I_∞ *factor.*
3. *For any* $\lambda \in \,]0,1]$ *there exists an uncountable family of pairwise non-isomorphic compact quantum groups such that* $\mathsf{L}^\infty(\mathbb{G})$ *is the injective factor of type* III_λ*.*
4. *There exists a family* $\{\mathbb{G}_s\}_{s\in]0,1[}$ *of compact quantum groups such that* $\{\mathsf{L}^\infty(\mathbb{G}_s)\}_{s\in]0,1[}$ *is a family of pairwise non-isomorphic injective factors of type* III_0*.*

Thank you for your attention