Why $\mathrm{B}(\ell^2)$ is not $L^\infty(\mathbb{G})$ and related topics

Seminarium Dyskretnej Analizy Harmonicznej

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The basics

Definition

A **compact quantum group** $\mathbb{G}$ is described by

- a von Neumann algebra $L^\infty(\mathbb{G})$,
- a unital $*$-homomorphism $\Delta\colon L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\,\bar{\otimes}\,L^\infty(\mathbb{G})$

(continuous in the σ-weak topology) such that

- $(\Delta\otimes\mathrm{id})\circ\Delta = (\mathrm{id}\otimes\Delta)\circ\Delta$,
- there exists a faithful state $\boldsymbol{h}$ on $L^\infty(\mathbb{G})$ such that

$$\forall\, x\in L^\infty(\mathbb{G})\ \ (\boldsymbol{h}\otimes\mathrm{id})\Delta(x) = \boldsymbol{h}(x)\mathbb{1} = (\mathrm{id}\otimes\boldsymbol{h})\Delta(x). \qquad (\heartsuit)$$

- The condition $(\heartsuit)$ determines $\boldsymbol{h}$ uniquely. We call this state the **Haar measure** of $\mathbb{G}$.

Definition

Let $\mathbb{G}$ be a compact quantum group. A **finite-dimensional unitary representation** of $\mathbb{G}$ is a unitary $U \in \mathrm{B}(\mathsf{H}) \otimes L^\infty(\mathbb{G})$ (with H a finite-dimensional Hilbert space) such that

$$(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13},$$

where

- $U_{12} = U \otimes \mathbb{1} \in \mathrm{B}(\mathsf{H}) \otimes L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})$,
- $U_{13} = (\mathrm{id} \otimes \mathrm{flip})(U_{12}) \in \mathrm{B}(\mathsf{H}) \otimes L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})$.

- We say that a representation $U \in \mathrm{B}(\mathsf{H}) \otimes L^\infty(\mathbb{G})$ is **irreducible** if $(T \otimes \mathbb{1})U = U(T \otimes \mathbb{1})$ implies $T = \lambda \mathbb{1}_{\mathsf{H}}$.
- Representations $U \in \mathrm{B}(\mathsf{H}) \otimes L^\infty(\mathbb{G})$ and $V \in \mathrm{B}(\mathsf{K}) \otimes L^\infty(\mathbb{G})$ are **equivalent** if there is a unitary $S \in \mathrm{B}(\mathsf{H}, \mathsf{K})$ such that $(S \otimes \mathbb{1})U = V(S \otimes \mathbb{1})$.

Matrix elements of irreps

- Let $U \in \mathrm{B}(\mathsf{H}) \otimes L^\infty(\mathbb{G})$ be a representation. Then any $\omega \in \mathrm{B}(\mathsf{H})^*$ defines $(\omega \otimes \mathrm{id})(U) \in L^\infty(\mathbb{G})$ which is called a **matrix element** or a **coefficient** of U.
- Typically we take $\omega(\cdot) = \langle\xi|\cdot|\eta\rangle$ for some vectors $\xi, \eta \in \mathsf{H}$.
- Choosing an orthonormal basis $\{\xi_1, \ldots, \xi_n\}$ of H yields $U_{i,j} = (\omega_{i,j} \otimes \mathrm{id})(U)$ where $\omega_{i,j} = \langle\xi_i|\cdot|\xi_j\rangle$.
- From now on we denote by $\mathrm{Irr}(\mathbb{G})$ the set of equivalence classes of irreps of $\mathbb{G}$. For each $\alpha \in \mathrm{Irr}(\mathbb{G})$ we fix $U^\alpha \in \alpha$. Then any orthonormal basis $\xi_1^\alpha, \ldots, \xi_{n_\alpha}^\alpha$ of the carrier Hilbert space H^α of U^α defines the matrix elements $U^\alpha_{i,j}$.

Theorem

$\mathrm{span}\bigl\{U^\alpha_{i,j} \,\big|\, \alpha \in \mathrm{Irr}(\mathbb{G}),\ i,j \in \{1, \ldots, n_\alpha\}\bigr\}$ *is σ-weakly dense in $L^\infty(\mathbb{G})$.*

THE ρ-OPERATORS

- For each $\alpha\in\mathrm{Irr}(\mathbb{G})$ let $V^\alpha=(j\otimes\mathrm{id})\bigl(U^{\alpha*}\bigr)\in B(\mathsf{H}^{\alpha*})\otimes L^\infty(\mathbb{G})$ ($j\colon B(\mathsf{H}^\alpha)\to B(\mathsf{H}^{\alpha*})$ maps T to the operator $\langle\psi|\mapsto\langle T^*\psi|$).
- Next we let $\rho_\alpha=\mathrm{const}\cdot j\bigl((\mathrm{id}\otimes\boldsymbol{h})\bigl(V^{\alpha*}V^\alpha\bigr)\bigr)$ with the constant chosen so that $\mathrm{Tr}(\rho_\alpha)=\mathrm{Tr}(\rho_\alpha^{-1})$.
- Note that ρ_α is positive.
- From now on for each $\alpha\in\mathrm{Irr}(\mathbb{G})$ we fix an orthonormal basis of H^α in which ρ_α is diagonal:

$$\rho_\alpha=\begin{bmatrix}\rho_{\alpha,1}&&\\&\ddots&\\&&\rho_{\alpha,n_\alpha}\end{bmatrix}$$

  and $\rho_{\alpha,1}\geqslant\cdots\geqslant\rho_{\alpha,n_\alpha}$.
- We have $\boldsymbol{h}\bigl(U^{\alpha}_{k,l}{}^{*}U^{\beta}_{i,j}\bigr)=\delta_{\alpha\beta}\frac{\delta_{ki}\rho_{\alpha,j}^{-1}\delta_{l,j}}{\mathrm{Tr}(\rho_\alpha)}$, so $\bigl\{U^\alpha_{i,j}\bigr\}$ are linearly independent.

The modular group and the scaling group

Theorem

There exist two σ-weakly continuous one-parameter groups $\sigma^{\boldsymbol{h}}$ and $\tau^{\mathbb{G}}$ of automorphisms of $L^\infty(\mathbb{G})$ such that

$$\tau^{\mathbb{G}}_t(U^\alpha_{i,j}) = \rho^{it}_{\alpha,i}U^\alpha_{i,j}\rho^{-it}_{\alpha,j}$$

$$\sigma^{\boldsymbol{h}}_t(U^\alpha_{i,j}) = \rho^{it}_{\alpha,i}U^\alpha_{i,j}\rho^{it}_{\alpha,j}$$

for all $\alpha \in \mathrm{Irr}(\mathbb{G})$, $i,j \in \{1,\dots,n_\alpha\}$ and $t \in \mathbb{R}$.

- Clearly the two groups commute.

What if?

Suppose that there is a compact quantum group $\mathbb{G}$ such that $L^\infty(\mathbb{G}) \cong B(\mathsf{H})$, where H is a Hilbert space such that $\dim \mathsf{H} > 1$.

- If H were finite-dimensional then $B(\mathsf{H})$ would be simple, but a finite dimensional $L^\infty(\mathbb{G})$ admits a character, so this is impossible.
- The case $\dim \mathsf{H} > \aleph_0$ is ruled out by the fact that there are no faithful normal states on $B(\mathsf{H})$ for non-separable H.
- Thus we are left with $\mathsf{H} \cong \ell^2$.
- We will show that this leads to a contradiction.

Step 1.

- Suppose $\mathbb{G}$ is a compact quantum group with $L^\infty(\mathbb{G}) \cong B(\mathsf{H})$.
- The state $\boldsymbol{h}$ cannot be a trace because there are no traces on $B(\mathsf{H})$.
- It is known that in this case ($\boldsymbol{h}$ not a trace) there exists $\alpha \in \mathrm{Irr}(\mathbb{G})$ with

$$(\rho_{\alpha,1}, \ldots, \rho_{\alpha,n_\alpha}) \neq (1, \ldots, 1).$$

- Let us assume that the set $\{\rho_{\alpha,1}, \ldots, \rho_{\alpha,n_\alpha}\}$ is invariant under taking inverses.

  If this doesn't hold we can show that the compact quantum group $\mathbb{G} \times \mathbb{G}$ has $\beta \in \mathrm{Irr}(\mathbb{G}\times\mathbb{G})$ such that ρ_β is non-trivial and $\{\rho_{\beta,1}, \ldots, \rho_{\beta,n_\beta}\} = \{\rho_{\beta,1}^{-1}, \ldots, \rho_{\beta,n_\beta}^{-1}\}$. Still $L^\infty(\mathbb{G} \times \mathbb{G}) = L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} L^\infty(\mathbb{G}) \cong B(\mathsf{H}) \mathbin{\overline{\otimes}} B(\mathsf{H}) \cong B(\mathsf{H})$.

Step 2.

- Let $\pi\colon L^\infty(\mathbb{G}) \to B(\mathsf{H})$ be the assumed isomorphism.
- The state $\boldsymbol{h}$ must be of the form

$$\boldsymbol{h}(x) = \operatorname{Tr}\big(A\pi(x)\big), \qquad x \in L^\infty(\mathbb{G})$$

  for some positive trace-class operator A on H with eigenvalues $q_1 > q_2 > \cdots > 0$.
- For each n let $\mathsf{H}(A = q_n)$ be the corresponding eigenspace, so that

$$\mathsf{H} = \bigoplus_{n=1}^{\infty} \mathsf{H}(A = q_n).$$

  Moreover, we have $\dim \mathsf{H}(A = q_n) < +\infty$ for all n.
- We have

$$\pi\big(\sigma_t^{\boldsymbol{h}}(x)\big) = A^{\mathrm{i}t}\pi(x)A^{-\mathrm{i}t}, \qquad x \in L^\infty(\mathbb{G}),\ t \in \mathbb{R}.$$

Step 3.

- There is a strictly positive self-adjoint operator B on H such that

$$\pi\bigl(\tau_t^{\mathbb{G}}(x)\bigr) = B^{\mathrm{i}t}\pi(x)B^{-\mathrm{i}t}, \qquad x \in L^\infty(\mathbb{G}),\ t \in \mathbb{R}$$

  (this is a consequence of Stone's theorem).
- The fact that the groups $\{\sigma_t^{\boldsymbol{h}}\}_{t\in\mathbb{R}}$ and $\{\tau_t^{\mathbb{G}}\}_{t\in\mathbb{R}}$ commute implies that A and B strongly commute.
- Hence for any n the operator B restricts to a positive operator on the finite-dimensional Hilbert space $\mathsf{H}(A = q_n)$.
- Let $\mu_{n,1} > \cdots > \mu_{n,P_n}$ be the complete list of eigenvalues of this restriction.
- We have

$$\mathsf{H} = \bigoplus_{n=1}^{\infty} \bigoplus_{p=1}^{P_n} \mathsf{H}(A = q_n) \cap \mathsf{H}(B = \mu_{n,p}).$$

Step 4.

- Claim: $\pi(U^\alpha_{k,1})$ maps $\mathsf{H}(A = q_n)$ into $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_n)$.
- Indeed: take $\xi \in \mathsf{H}(A = q_n)$. Then

$$\begin{aligned} A^{\mathrm{i}t}\pi(U^\alpha_{k,1})\xi &= A^{\mathrm{i}t}\pi(U^\alpha_{k,1})A^{-\mathrm{i}t}A^{\mathrm{i}t}\xi = \pi\big(\sigma^{\boldsymbol{h}}_t(U^\alpha_{k,1})\big)q_n^{\mathrm{i}t}\xi \\ &= \pi\big(\rho^{\mathrm{i}t}_{\alpha,k}U^\alpha_{k,1}\rho^{\mathrm{i}t}_{\alpha,1}\big)q_n^{\mathrm{i}t}\xi = (\rho_{\alpha,k}\rho_{\alpha,1}q_n)^{\mathrm{i}t}\pi(U^\alpha_{k,1})\xi. \end{aligned}$$

- Claim: $\pi(U^\alpha_{k,1})$ maps $\mathsf{H}(B = \mu_{n,p})$ into $\mathsf{H}(B = \rho_{\alpha,k}\rho^{-1}_{\alpha,1}\mu_{n,p})$.
- Indeed: take $\eta \in \mathsf{H}(B = \mu_{n,p})$. Then

$$\begin{aligned} B^{\mathrm{i}t}\pi(U^\alpha_{k,1})\eta &= B^{\mathrm{i}t}\pi(U^\alpha_{k,1})B^{-\mathrm{i}t}B^{\mathrm{i}t}\eta = \pi\big(\tau^{\mathbb{H}}_t(U^\alpha_{k,1})\big)\mu^{\mathrm{i}t}_{n,p}\eta \\ &= \pi\big(\rho^{\mathrm{i}t}_{\alpha,k}U^\alpha_{k,1}\rho^{-\mathrm{i}t}_{\alpha,1}\big)\mu^{\mathrm{i}t}_{n,p}\eta = \big(\rho_{\alpha,k}\rho^{-1}_{\alpha,1}\mu_{n,p}\big)^{\mathrm{i}t}\pi(U^\alpha_{k,1})\eta. \end{aligned}$$

- Let ζ be a non-zero element of $\mathsf{H}(A = q_1) \cap \mathsf{H}(B = \mu_{1,P_1})$.
  We will show that $\pi(U^\alpha_{k,1})\zeta = 0$ for all $k \in \{1, \dots, n_\alpha\}$.

Step 4. (continued)

- By the previous claims we have

$$\pi(U^\alpha_{k,1})\zeta \in \mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) \cap \mathsf{H}(B = \rho_{\alpha,k}\rho_{\alpha,1}^{-1}\mu_{1,P_1}).$$

- If $\rho_{\alpha,k} = \rho_{\alpha,1}$ then $\rho_{\alpha,k}\rho_{\alpha,1}q_1 = \rho_{\alpha,1}^2 q_1 > q_1 = \|A\|$, so $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \{0\}$ and consequently $\pi(U^\alpha_{k,1})\zeta = 0$.
- If $\rho_{\alpha,k} < \rho_{\alpha,1}$ then first of all

$$\rho_{\alpha,k}\rho_{\alpha,1}q_1 \geqslant \bigl(\min_i\{\rho_{\alpha,i}\}\bigr)\rho_{\alpha,1}q_1 = \rho_{\alpha,1}^{-1}\rho_{\alpha,1}q_1 = q_1$$

(invariance of $\{\rho_{\alpha,1},\dots,\rho_{\alpha,n_\alpha}\}$ under taking inverses!). Thus

$$\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \mathsf{H}(A = q_1) \quad \text{or} \quad \mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \{0\}.$$

Clearly, if $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \{0\}$ then $\pi(U^\alpha_{k,1})\zeta = 0$.

Step 4. (continued further)

- We have $\pi(U^\alpha_{k,1})\zeta \in \mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) \cap \mathsf{H}(B = \rho_{\alpha,k}\rho_{\alpha,1}^{-1}\mu_{1,P_1})$ and $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \mathsf{H}(A = q_1)$ or $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \{0\}$.
- What happens if $\mathsf{H}(A = \rho_{\alpha,k}\rho_{\alpha,1}q_1) = \mathsf{H}(A = q_1)$?
- In this case $\rho_{\alpha,k}$ must be $\rho_{\alpha,1}^{-1}$, so

$$\rho_{\alpha,k}\rho_{\alpha,1}^{-1}\mu_{1,P_1} = \rho_{\alpha,1}^{-2}\mu_{1,P_1} < \mu_{1,P_1} = \min \operatorname{Sp}\bigl(B\big|_{\mathsf{H}(A=q_1)}\bigr).$$

Consequently $\mathsf{H}\bigl(B = \rho_{\alpha,k}\rho_{\alpha,1}^{-1}\mu_{1,P_1}\bigr) = \{0\}$ and

$$\pi(U^\alpha_{k,1})\zeta \in \mathsf{H}(A = q_1) \cap \mathsf{H}\bigl(B = \rho_{\alpha,k}\rho_{\alpha,1}^{-1}\mu_{1,P_1}\bigr) = \{0\}.$$

In particular $\pi(U^\alpha_{k,1})\zeta = 0$.

Step 5.

- We have shown that there is a non-zero $\zeta \in \mathsf{H}$ with

$$\pi(U^\alpha_{k,1})\zeta, \qquad k = 1, \ldots, n_\alpha.$$

- But $U^\alpha = \begin{bmatrix} U^\alpha_{1,1} & \cdots & U^\alpha_{1,n_\alpha} \\ \vdots & \ddots & \vdots \\ U^\alpha_{n_\alpha,1} & \cdots & U^\alpha_{n_\alpha,n_\alpha} \end{bmatrix}$ is a unitary matrix, so

$$0 \neq \zeta = \sum_{k=1}^{n_\alpha} \pi(U^\alpha_{k,1})^* \pi(U^\alpha_{k,1})\zeta = 0.$$

- This contradiction shows that the existence of $\mathbb{G}$ such that $L^\infty(\mathbb{G}) \cong B(\mathsf{H})$ is impossible.

$\square$

Remarks

1. The proof can be tweaked to obtain

Theorem (J. Krajczok & P.M.S.)

There does not exist a compact quantum group $\mathbb{G}$ *such that* $L^\infty(\mathbb{G}) \cong \mathsf{N} \oplus B(\mathsf{H})$ *with* N *an arbitrary von Neumann algebra or the zero vector space and* H *of infinite dimension.*

Remarks

2. Similar techniques yield the following

Theorem (A. Chirvasitu, J. Krajczok & P.M.S.)

Let $\mathbb{G}$ be a compact quantum group such that the C^*-algebra $C(\mathbb{G})$ fits into the exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{N} \mathcal{K}(\mathsf{H}_i) \longrightarrow C(\mathbb{G}) \longrightarrow C(X) \longrightarrow 0$$

with X a compact space. The $\mathbb{G}$ is finite $(\dim C(\mathbb{G}) < +\infty)$.

3. It follows that the Podleś spheres and the quantum disk do not admit a structure of a compact quantum group.

Theorem (J. Krajczok & M. Wasilewski)

Let $q \in]-1,1[\setminus\{0\}$ and $\nu \in \mathbb{R}\setminus\{0\}$ and consider the action α^{ν} of $\mathbb{Q}$ with discrete topology on $\mathrm{SU}_q(2)$ given by

$$\alpha^{\nu}_r(x) = \tau^{\mathrm{SU}_q(2)}_{\nu r}(x), \qquad x \in L^{\infty}\big(\mathrm{SU}_q(2)\big),\ r \in \mathbb{Q}.$$

Let $\mathbb{H}_{\nu,q}$ be the corresponding bicrossed product:

$$\mathbb{H}_{\nu,q} = \mathbb{Q} \bowtie \mathrm{SU}_q(2).$$

Then

1. $\mathbb{H}_{\nu,q}$ is a compact quantum group,
2. $\mathbb{H}_{\nu,q}$ is coamenable and hence $L^{\infty}(\mathbb{H}_{\nu,q})$ is injective,
3. if $\nu \log|q| \notin \pi\mathbb{Q}$ then $L^{\infty}(\mathbb{H}_{\nu,q})$ is the injective factor of type II_{∞},
4. the spectrum of the modular operator for the Haar measure $\boldsymbol{h}_{\nu,q}$ of $\mathbb{H}_{\nu,q}$ is $\{0\} \cup q^{2\mathbb{Z}}$.

- Let $\big((\nu_n, q_n)\big)_{n\in\mathbb{N}}$ be a sequence of parameters as described above ($\nu_n \log|q_n| \notin \pi\mathbb{Q}$ for all n) and consider the compact quantum group

$$\mathbb{G} = \bigtimes_{n=1}^{\infty} \mathbb{H}_{\nu_n,q_n}.$$

- In particular $L^\infty(\mathbb{G}) = \bigotimes_{n=1}^{\infty} L^\infty(\mathbb{H}_{\nu_n,q_n})$.

EXAMPLE

If the sequence $\big((\nu_n, q_n)\big)_{n\in\mathbb{N}}$ is constant then $L^\infty(\mathbb{G})$ is the injective factor of type III_{q^2} with separable predual.

- $T\big(L^\infty(\mathbb{G})\big) = \frac{\pi}{\log|q|}\mathbb{Z}$,
- $S\big(L^\infty(\mathbb{G})\big) = \{0\} \cup |q|^{2\mathbb{Z}}$.

EXAMPLE

If there are two subsequences $(q_{n_{1,p}})_{p\in\mathbb{N}}$ and $(q_{n_{2,p}})_{p\in\mathbb{N}}$ such that

$$\{n_{1,p}\,|\,p\in\mathbb{N}\}\cap\{n_{2,p}\,|\,p\in\mathbb{N}\}=\varnothing$$

and

$$q_{n_{1,p}}\xrightarrow[p\to\infty]{}r_1,\quad q_{n_{2,p}}\xrightarrow[p\to\infty]{}r_2$$

for some $r_1,r_2\in\,]-1,1[\,\setminus\{0\}$ such that $\frac{\pi}{\log|r_1|}\mathbb{Z}\cap\frac{\pi}{\log|r_2|}\mathbb{Z}=\{0\}$ then $L^\infty(\mathbb{G})$ is the injective factor of type III_1 with separable predual.

- $T\big(L^\infty(\mathbb{G})\big)=\{0\}$,
- $S\big(L^\infty(\mathbb{G})\big)=\mathbb{R}_{\geqslant 0}$.

THEOREM (J. KRAJCZOK & P.M.S.)

There exist a family $\{\mathbb{G}_s\}_{s\in]0,1[}$ of compact quantum groups such that the von Neumann algebras $\{L^\infty(\mathbb{G}_s)\}_{s\in]0,1[}$ are pairwise non-isomorphic factors of type III_0.

- $T\bigl(L^\infty(\mathbb{G}_s)\bigr) \supset \mathbb{Q}$,
- defining

$$t_s = \sum_{p=1}^{\infty} \frac{\lfloor p^{1-s} \rfloor}{p!}, \qquad s \in]0,1[$$

we have

$$\Bigl(t_{s'} \in T\bigl(L^\infty(\mathbb{G}_s)\bigr) \Bigr) \iff \Bigl(s' > s \Bigr).$$

- For each $\lambda \in]0,1]$ there exists uncountably many pairwise non-isomorphic compact quantum groups with $L^\infty(\mathbb{G})$ the injective factor of type III_λ.
- These compact quantum groups are constructed as bicrossed products $\Gamma \bowtie \mathop{\times}\limits_{n=1}^{\infty} \mathbb{H}_{\nu_n,q_n}$ with Γ a subgroup of $\mathbb{R}$ (taken with discrete topology) acting by the scaling automorphisms.
- We distinguish between them using the following invariants:
  - $T^\tau(\mathbb{G}) = \left\{t \in \mathbb{R} \,\middle|\, \tau^{\mathbb{G}}_t = \mathrm{id}\right\}$,
  - $T^\tau_{\mathrm{Inn}}(\mathbb{G}) = \left\{t \in \mathbb{R} \,\middle|\, \tau^{\mathbb{G}}_t \in \mathrm{Inn}(L^\infty(\mathbb{G}))\right\}$,
  - $T^\tau_{\overline{\mathrm{Inn}}}(\mathbb{G}) = \left\{t \in \mathbb{R} \,\middle|\, \tau^{\mathbb{G}}_t \in \overline{\mathrm{Inn}}(L^\infty(\mathbb{G}))\right\}$.

Full list of invariants

Definition

Let $\mathbb{G}$ be a locally compact quantum group. We define

$$
\begin{aligned}
T^{\tau}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\tau^{\mathbb{G}}_t=\mathrm{id}\bigr\},\\
T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\tau^{\mathbb{G}}_t\in\mathrm{Inn}\bigl(L^{\infty}(\mathbb{G})\bigr)\bigr\},\\
T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\tau^{\mathbb{G}}_t\in\overline{\mathrm{Inn}}\bigl(L^{\infty}(\mathbb{G})\bigr)\bigr\},\\
T^{\sigma}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\sigma^{\varphi}_t=\mathrm{id}\bigr\},\\
T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\sigma^{\varphi}_t\in\mathrm{Inn}\bigl(L^{\infty}(\mathbb{G})\bigr)\bigr\},\\
T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\sigma^{\varphi}_t\in\overline{\mathrm{Inn}}\bigl(L^{\infty}(\mathbb{G})\bigr)\bigr\},\\
\mathrm{Mod}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\delta^{\mathrm{i}t}=\mathbb{1}\bigr\},
\end{aligned}
$$

where δ is the modular element of $\mathbb{G}$.

- The sets $T^{\circ}_{\bullet}(\mathbb{G})$ are subgroups of $\mathbb{R}$ and are isomorphism invariants of the quantum group $\mathbb{G}$.
- $T^{\tau}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}})$.
- $T^{\bullet}(\mathbb{G})$, $T^{\bullet}_{\overline{\mathrm{Inn}}}(\mathbb{G})$, and $\mathrm{Mod}(\mathbb{G})$ are closed.
- We would obtain the same groups $T^{\sigma}(\mathbb{G})$, $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G})$, and $T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G})$ if we chose the right Haar measure instead of the left one.
- $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G})$ is equal to the Connes' invariant $T(L^{\infty}(\mathbb{G}))$. Consequently, $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G})$ depends only on the von Neumann algebra $L^{\infty}(\mathbb{G})$. It is also the case for $T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G})$.

PROPOSITION

For any locally compact quantum group $\mathbb{G}$ we have

$$\begin{aligned} T^{\sigma}(\mathbb{G}) &= T^{\tau}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}), \\ T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}) &= T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}), \\ T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}) &= T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}), \\ \operatorname{Mod}(\mathbb{G}) \cap \operatorname{Mod}(\widehat{\mathbb{G}}) &\subset \tfrac{1}{2}\, T^{\tau}(\mathbb{G}). \end{aligned}$$

- The first equality above together with $T^{\tau}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}})$ reduces the list to 11 (invariants $T^{\sigma}(\mathbb{G})$, $T^{\sigma}(\widehat{\mathbb{G}})$ and $T^{\tau}(\widehat{\mathbb{G}})$ are determined by the remaining ones).
- If $\mathbb{G}$ is compact then $\operatorname{Mod}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathbb{R}$.
- If additionally $L^{\infty}(\mathbb{G})$ is semifinite then $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = \mathbb{R}$.

Example: the quantum $E(2)$ group

With $\mathbb{G} = \mathrm{E}_q(2)$ for some $q \in \,]0,1[$ we have

$$T^\tau(\mathbb{G}) = T^\tau_{\mathrm{Inn}}(\mathbb{G}) = T^\tau_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\sigma(\mathbb{G}) = T^\tau(\widehat{\mathbb{G}}) = T^\sigma(\widehat{\mathbb{G}}) = \mathrm{Mod}(\widehat{\mathbb{G}}) = \tfrac{\pi}{\log q}\mathbb{Z},$$

$$T^\sigma_{\mathrm{Inn}}(\mathbb{G}) = T^\sigma_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\tau_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^\tau_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^\sigma_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^\sigma_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathrm{Mod}(\mathbb{G}) = \mathbb{R}.$$

Example: quantum "$az+b$" groups

Let $\mathbb{G}$ be the quantum "$az+b$" group for the deformation parameter q in one of the three cases:

1. $q = e^{\frac{2\pi i}{N}}$ with $N = 6, 8, \ldots,$
2. $q \in \,]0, 1[,$
3. $q = e^{1/\rho}$ with $\operatorname{Re}\rho < 0$, $\operatorname{Im}\rho = \frac{N}{2\pi}$ with $N = \pm 2, \pm 4, \ldots.$

Then

$$T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^{\sigma}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathbb{R},$$

$$T^{\tau}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}}) = T^{\sigma}(\mathbb{G}) = T^{\sigma}(\widehat{\mathbb{G}}) = \mathrm{Mod}(\mathbb{G}) = \mathrm{Mod}(\widehat{\mathbb{G}}) = \begin{cases} \{0\} & \text{in cases 1 and 3} \\ \frac{\pi}{\log q}\mathbb{Z} & \text{in case 2} \end{cases}.$$

Example: U_F^+

Let $\mathbb{G}$ be the quantum group U_F^+. Then $L^\infty(\mathbb{G})$ is a full factor so $\mathrm{Inn}(L^\infty(\mathbb{G})) = \overline{\mathrm{Inn}}(L^\infty(\mathbb{G}))$ (Vaes).

- $\mathbb{G}$ is compact, so $\mathrm{Mod}(\mathbb{G}) = T^\tau_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^\tau_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^\sigma_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^\sigma_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathbb{R}$.
- If $\mathbb{G}$ is not of Kac type ($F^*F = \mathbb{1}$) then

$$T^\tau_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\tau_{\mathrm{Inn}}(\mathbb{G}) = T^\tau(\mathbb{G}) = \bigcap_{\Lambda\in\mathrm{Sp}(F^*F\otimes(F^*F)^{-1})\setminus\{1\}} \frac{2\pi}{\log(\Lambda)}\mathbb{Z},$$

  while $\mathrm{Mod}(\widehat{\mathbb{G}}) = \bigcap_{\Lambda\in\mathrm{Sp}(F^*F)\setminus\{\lambda^{-1}\}} \frac{2\pi}{\log\lambda+\log(\Lambda)}\mathbb{Z}$, where $\lambda = \sqrt{\frac{\mathrm{Tr}((F^*F)^-1)}{\mathrm{Tr}(F^*F)}}$.
- If $\mathbb{G}$ is not of Kac type then $L^\infty(\mathbb{G})$ is a type III_μ factor for some $\mu \in]0,1]$ and $T^\sigma_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\sigma_{\mathrm{Inn}}(\mathbb{G}) = \frac{2\pi}{\log\mu}\mathbb{Z}$ (otherwise $T^\sigma_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\sigma_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$).

Remarks

- In the course of constructing families of $L^\infty(\mathbb{G})$ isomorphic to factors of type III, for any subgroup Γ of $\mathbb{R}$ we constructed second countable compact quantum group $\mathbb{K}$ such that $T^\tau_{\mathrm{Inn}}(\mathbb{K})=\Gamma$.
- The equality $T^\tau(\mathrm{U}_F^+)=T^\tau_{\mathrm{Inn}}(\mathrm{U}_F^+)$ says that the compact quantum group U_F^+ belongs to the class for which the following statement is true:

Conjecture (∗)

If $\mathbb{G}$ is a second countable compact quantum group and $T^\tau_{\mathrm{Inn}}(\mathbb{G})=\mathbb{R}$ then $\mathbb{G}$ is of Kac type.

- We were able to prove that this conjecture is true for many compact quantum groups including duals of second countable type I discrete quantum groups (e.g. q-deformations of compact semisimple Lie groups).

Example: q-deformations

Let G be a compact semisimple Lie group with root system Φ and let $q \in \,]0,1[$.

- Since G_q is compact we again have

$$\operatorname{Mod}(G_q) = T^{\tau}_{\mathrm{Inn}}(\widehat{G_q}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\widehat{G_q}) = T^{\sigma}_{\mathrm{Inn}}(\widehat{G_q}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\widehat{G_q}) = \mathbb{R}.$$

- Furthermore $T^{\sigma}_{\mathrm{Inn}}(G_q) = T^{\sigma}_{\overline{\mathrm{Inn}}}(G_q) = \mathbb{R}$ because $\mathrm{C}(G_q)$ is a C*-algebra of type I.
- We have $T^{\tau}(G_q) = \frac{\pi}{\log q}\mathbb{Z}$ and

$$T^{\tau}_{\mathrm{Inn}}(G_q) = T^{\tau}_{\overline{\mathrm{Inn}}}(G_q) = \operatorname{Mod}(\widehat{G_q}) = \frac{\pi}{\Upsilon_{\Phi}\log q}\mathbb{Z},$$

where Υ_{Φ} is a positive integer determined by Lie-theoretic data (see next two slides).

Example: q-deformations (continued)

- Let $\Phi=\Phi_1\cup\dots\cup\Phi_l$ be the decomposition of Φ into irreducible parts. Then

$$\Upsilon_\Phi=\gcd(\Upsilon_{\Phi_1},\dots,\Upsilon_{\Phi_l}).$$

- We have

type	**group**	**range of n**	Υ_Φ	$T^\tau_{\mathrm{Inn}}(G_q)$
A_n	$\mathrm{SU}(n+1)$	$n\geqslant 1$ odd	1	$\frac{\pi}{\log q}\mathbb{Z}$
		$n\geqslant 1$ even	2	$\frac{\pi}{2\log q}\mathbb{Z}$
B_n	$\mathrm{Spin}(2n+1)$	$n\geqslant 2$ odd	1	$\frac{\pi}{\log q}\mathbb{Z}$
		$n\geqslant 2$ even	2	$\frac{\pi}{2\log q}\mathbb{Z}$
C_n	$\mathrm{Sp}(2n)$	$n\geqslant 3$	2	$\frac{\pi}{2\log q}\mathbb{Z}$
D_n	$\mathrm{Spin}(2n)$	$n\geqslant 4$ odd	2	$\frac{\pi}{2\log q}\mathbb{Z}$
		$n\geqslant 4$ even	1	$\frac{\pi}{\log q}\mathbb{Z}$

Example: q-deformations (continued)

- And for the exceptional cases we have
  - type E_6: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{2\log q}\mathbb{Z}$,
  - type E_7: $\Upsilon_\Phi = 1$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{\log q}\mathbb{Z}$,
  - type E_8: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{2\log q}\mathbb{Z}$,
  - type F_4: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{2\log q}\mathbb{Z}$,
  - type G_2: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(\mathbb{G}_q) = \frac{\pi}{2\log q}\mathbb{Z}$.

Special case

- Consider the compact quantum group $\mathrm{SU}_q(3)$.
- Then $\Upsilon_\Phi = 2$, so
$$T^\tau_{\mathrm{Inn}}\big(\mathrm{SU}_q(3)\big) = \tfrac{\pi}{2\log q}\mathbb{Z},$$
while $T^\tau\big(\mathrm{SU}_q(3)\big) = \tfrac{\pi}{\log q}\mathbb{Z}$.
- This means that there are non-trivial inner scaling automorphisms.
- $\mathrm{SU}_q(3)$ does not have non-trivial one-dimensional representations, so these scaling automorphisms are not implemented by a group-like element.

Proposition

Let G be such that $\Upsilon_\Phi = 2$. Then a unitary implementing the scaling automorphism for $t = \frac{\pi}{2\log q}$ does not belong to $\mathrm{C}(G_q)$. In particular, the restriction of this automorphism to $\mathrm{C}(G_q)$ is not inner.

AND NOW FOR SOMETHING COMPLETELY DIFFERENT

PROPOSITION

Let Γ be a discrete group. Then the following are equivalent:

1. Γ is i.c.c.,
2. $L(\Gamma)$ is a factor,
3. $\Delta_{\hat{\Gamma}}^{(n)}(L(\Gamma))' \cap \underbrace{L(\Gamma)\,\overline{\otimes}\cdots\overline{\otimes}\,L(\Gamma)}_{n+1} = \mathbb{C}\mathbb{1}$ for some $n \in \mathbb{N}$,
4. $\Delta_{\hat{\Gamma}}^{(n)}(L(\Gamma))' \cap \underbrace{L(\Gamma)\,\overline{\otimes}\cdots\overline{\otimes}\,L(\Gamma)}_{n+1} = \mathbb{C}\mathbb{1}$ for all $n \in \mathbb{N}$.

PROPOSITION

Let $\mathbb{G}$ be a locally compact quantum group and assume that

$$\Delta_{\mathbb{G}}^{(n)}\big(L^{\infty}(\mathbb{G})\big)' \cap \underbrace{L^{\infty}(\mathbb{G})\,\overline{\otimes}\,\cdots\,\overline{\otimes}\,L^{\infty}(\mathbb{G})}_{n+1} = \mathbb{C}\mathbb{1}$$

for some $n \in \mathbb{N}$. Then $L^{\infty}(\mathbb{G})$ is a factor.

DEFINITION

Let $\mathbb{\Gamma}$ be a discrete quantum group. We say that $\mathbb{\Gamma}$ is n-i.c.c. if

$$\Delta_{\widehat{\mathbb{\Gamma}}}^{(n)}\big(L^{\infty}(\widehat{\mathbb{\Gamma}})\big)' \cap \underbrace{L^{\infty}(\widehat{\mathbb{\Gamma}})\,\overline{\otimes}\,\cdots\,\overline{\otimes}\,L^{\infty}(\widehat{\mathbb{\Gamma}})}_{n+1} = \mathbb{C}\mathbb{1}.$$

Proposition

Let $\mathbb{\Gamma}$ be a discrete quantum group. If $\mathbb{\Gamma}$ is n-i.c.c. for some n then $\mathbb{\Gamma}$ is m-i.c.c. for all natural $m \leqslant n$.

Theorem

Let $\mathbb{G}$ be a second countable compact quantum group whose dual is 1-i.c.c. Then conjecture $(*)$ holds for $\mathbb{G}$.

- Recall that $\mathrm{Irr}(\mathrm{U}_F^+) = \mathbb{Z}_+ \star \mathbb{Z}_+$ with the two copies of $\mathbb{Z}_+$ generated by the class α of the defining representation and $\beta = \overline{\alpha}$.
- For $x \in \mathbb{Z}_+ \star \mathbb{Z}_+$ put

$$D_{x,n} = \begin{cases} \|\rho_x^2 - \mathbb{1}\| \frac{\|\rho_x\|^{2(n+1)} - 1}{\|\rho_x\|^2 - 1} & \rho_x \neq \mathbb{1} \\ 0 & \rho_x = \mathbb{1} \end{cases}.$$

- Let $D_n = \max\{D_{\alpha\beta,n}, D_{\beta\alpha,n}, D_{\alpha^2\beta,n}\}$.

THEOREM

If $D_n < 1 - \frac{1}{\sqrt{2}}$ and $\frac{2(7-4D_n)D_n}{2(1-D_n)^2-1} < \frac{1}{\sqrt{n+1}}$ then $\widehat{\mathrm{U}_F^+}$ is n-i.c.c.

Theorem

Take $n \in \mathbb{N}$ and write $c = \max\Bigl\{\|\lambda F^*F - \mathbb{1}\|, \bigl\|(\lambda F^*F)^{-1} - \mathbb{1}\bigr\|\Bigr\}$, where $\lambda = \sqrt{\frac{\mathrm{Tr}((F^*F)^{-1})}{\mathrm{Tr}(F^*F)}}$. If

$$\sqrt{n}(n+1)c(2+c)(1+c)^{4+6n} < \tfrac{1}{72}$$

then $\widehat{\mathrm{U}_F^+}$ is n-i.c.c.

Post-doc position in Warsaw

- A position for one year starting March 2024 is being announced.
- Please e-mail me if you are interested.

Thank you for your attention