RESTRICTING REPRESENTATIONS TO A NORMAL SUBGROUP

COMPACT QUANTUM GROUPS
ALFRIED KRUPP WISSENSCHAFTSKOLLEG GREIFSWALD

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A.H. Clifford (1937), also G. Frobenius (1898): let

- G be a group,
- $K \subset G$ a normal subgroup,
- $U\colon G \longrightarrow \mathrm{B}(\mathscr{H}_U)$ — a finite dimensional irreducible representation.

Restricting U to K we obtain a finite dimensional representation $U\big|_K$ of K on $\mathscr{H}_U$.

Theorem

1. *The representation $U\big|_K$ is either irreducible or fully reducible,*
2. *in case $U\big|_K$ is reducible, its irreducible components constitute precisely one orbit of the action of G on $\mathrm{Irr}\,K$, each entering $U\big|_K$ with the same multiplicity.*

- G acts on $\mathrm{Irr}\,K$ in the following way:

$$(V \cdot g)(k) = V(gkg^{-1}), \qquad V \in \mathrm{Irr}\,K,\ g \in G,\ k \in K.$$

- $\mathbb{G}$ will denote a compact quantum group.
- A **representation** of $\mathbb{G}$ on a Hilbert space $\mathscr{H}$ is a unitary $u \in \mathrm{B}(\mathscr{H}) \otimes \mathrm{C}(\mathbb{G})$ such that

$$(\mathrm{id} \otimes \Delta)u = u_{12}u_{13}$$

or in other words

$$u = \begin{bmatrix} u_{1,1} & \dots & u_{1,n} \\ \vdots & \ddots & \vdots \\ u_{n,1} & \dots & u_{n,n} \end{bmatrix}$$

with $\Delta(u_{i,j}) = \sum_{k=1}^{n} u_{i,k} \otimes u_{k,j}$.

- All representations are fully reducible.

- Let $\{u^\alpha\}_{\alpha\in\mathrm{Irr}\,\mathbb{G}}$ be a complete collection of representatives of equivalence classes of irreps of $\mathbb{G}$,
- define

$$\mathrm{W}^{\mathbb{G}} = \bigoplus_{\alpha\in\mathrm{Irr}\,\mathbb{G}} u^\alpha \in \ell^\infty(\widehat{\mathbb{G}}) \bar{\otimes} L^\infty(\mathbb{G}),$$

where $\ell^\infty(\widehat{\mathbb{G}}) = \bigoplus_{\alpha\in\mathrm{Irr}\,\mathbb{G}} \mathrm{B}(\mathscr{H}_\alpha)$.

FACT

Any representation of $\mathbb{G}$ is of the form

$$u^\pi = (\pi\otimes\mathrm{id})\mathrm{W}^{\mathbb{G}}$$

for a unique representation of $\pi\colon\ \ell^\infty(\widehat{\mathbb{G}}) \longrightarrow \mathrm{B}(\mathscr{H}_\pi)$.

- $\ell^\infty(\widehat{\mathbb{G}})$ has a coproduct $\widehat{\Delta}$ such that $(\widehat{\Delta}\otimes\mathrm{id})\mathrm{W}^{\mathbb{G}} = \mathrm{W}^{\mathbb{G}}_{23}\mathrm{W}^{\mathbb{G}}_{13}$.

- We sometimes speak about $\pi\colon\ \ell^\infty(\widehat{\mathbb{G}}) \longrightarrow \mathrm{B}(\mathscr{H}_\pi)$ as a representation of $\mathbb{G}$.
- The irreducible representations correspond to projections

$$\ell^\infty(\widehat{\mathbb{G}}) = \bigoplus_{\alpha\in\operatorname{Irr}\mathbb{G}} \mathrm{B}(\mathscr{H}_\alpha) \longrightarrow \mathrm{B}(\mathscr{H}_{\alpha_0})$$

  for some $\alpha_0 \in \operatorname{Irr}\mathbb{G}$.
- Irreps of $\mathbb{G}$ are thus in bijection with the set of minimal central projections in $\ell^\infty(\widehat{\mathbb{G}})$. For $\alpha \in \operatorname{Irr}\mathbb{G}$ we write p_α for the corresponding projection.
- Summary of notation:

$$\pi\colon\ \ell^\infty(\widehat{\mathbb{G}}) \longrightarrow \mathrm{B}(\mathscr{H}_\pi), \quad u^\pi = (\pi \otimes \mathrm{id})\mathrm{W}^{\mathbb{G}}, \quad p_\pi \in \ell^\infty(\widehat{\mathbb{G}}).$$

- $\mathbb{K}$ is a closed quantum subgroup of $\mathbb{G}$ if $\ell^\infty(\widehat{\mathbb{K}})$ is embedded as a subalgebra of $\ell^\infty(\widehat{\mathbb{G}})$ in a $\widehat{\Delta}$-preserving way.
- If $\pi\colon \ell^\infty(\widehat{\mathbb{G}}) \longrightarrow \mathrm{B}(\mathscr{H}_\pi)$ is a representation then the **restriction of π to $\mathbb{K}$** is the representation of $\mathbb{K}$ corresponding to

$$\pi\big|_{\ell^\infty(\widehat{\mathbb{K}})}\colon \ell^\infty(\widehat{\mathbb{K}}) \to \mathrm{B}(\mathscr{H}_\pi).$$

- $\mathbb{K}$ is **normal** if $\mathrm{W}^{\mathbb{G}}\big(\ell^\infty(\widehat{\mathbb{K}}) \otimes \mathbb{1}\big)\mathrm{W}^{\mathbb{G}*} \subset \ell^\infty(\widehat{\mathbb{K}}) \bar{\otimes} L^\infty(\mathbb{G})$.

THEOREM

If $\mathbb{K}$ is normal then

$$\ell^\infty(\widehat{\mathbb{K}}) \ni x \longmapsto \mathrm{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{G}*} \in \ell^\infty(\widehat{\mathbb{K}}) \bar{\otimes} L^\infty(\mathbb{G}).$$

restricts to an action of $\mathbb{G}$ on $\mathscr{Z}(\ell^\infty(\widehat{\mathbb{K}}))$.

Proof.

Alternative way of expressing normality of $\mathbb{K}$ is

$$\mathrm{W}^{\mathbb{G}*}(\ell^\infty(\widehat{\mathbb{K}})\otimes\mathbb{1})\mathrm{W}^{\mathbb{G}}\subset\ell^\infty(\widehat{\mathbb{K}})\,\bar{\otimes}\,L^\infty(\mathbb{G}).$$

Take $y\in\ell^\infty(\widehat{\mathbb{K}})$. For $x\in\mathscr{Z}(\ell^\infty(\widehat{\mathbb{K}}))$ we have

$$\begin{aligned}\mathrm{W}^{\mathbb{G}}(x\otimes\mathbb{1})\mathrm{W}^{\mathbb{G}*}(y\otimes\mathbb{1})&=\mathrm{W}^{\mathbb{G}}(x\otimes\mathbb{1})\mathrm{W}^{\mathbb{G}*}(y\otimes\mathbb{1})\mathrm{W}^{\mathbb{G}}\mathrm{W}^{\mathbb{G}*}\\&=\mathrm{W}^{\mathbb{G}}\mathrm{W}^{\mathbb{G}*}(y\otimes\mathbb{1})\mathrm{W}^{\mathbb{G}}(x\otimes\mathbb{1})\mathrm{W}^{\mathbb{G}*}\\&=(y\otimes\mathbb{1})\mathrm{W}^{\mathbb{G}}(x\otimes\mathbb{1})\mathrm{W}^{\mathbb{G}*}\end{aligned}$$

which means that the left leg of $\mathrm{W}^{\mathbb{G}}(x\otimes\mathbb{1})\mathrm{W}^{\mathbb{G}*}$ belongs to $\mathscr{Z}(\ell^\infty(\widehat{\mathbb{K}}))$. The fact that $x\longmapsto\mathrm{W}^{\mathbb{G}}(x\otimes\mathbb{1})\mathrm{W}^{\mathbb{G}*}$ is an action of $\mathbb{G}$ on $\mathscr{Z}(\ell^\infty(\widehat{\mathbb{K}}))$ is easily checked. □

- If $\mathbb{K}$ is a normal closed quantum subgroup of $\mathbb{K}$ then $\mathbb{G}$ acts of $\mathscr{Z}(\ell^\infty(\widehat{\mathbb{K}}))$. Call this action $\alpha\colon\ \ell^\infty(\mathbb{N})\longrightarrow\ell^\infty(\mathbb{N})\,\bar{\otimes}\,L^\infty(\mathbb{G})$.
- $\mathscr{Z}(\ell^\infty(\widehat{\mathbb{K}}))$ is isomorphic to $\ell^\infty(\mathbb{N})$.
- Let $(e_i)_{i\in\mathbb{N}}$ be the "standard basis" of $\ell^\infty(\mathbb{N})$. We can express the action α as

$$\alpha(e_j)=\sum_{i=1}^{\infty}e_i\otimes s_{i,j}$$

  for some elements $\{s_{i,j}\}_{i,j\in\mathbb{N}}$ of $L^\infty(\mathbb{G})$ (s_i's are projections and the sum is strongly convergent).
- Easy to see form unitality of α that

$$\sum_{j=1}^{\infty}s_{i,j}=\mathbb{1},\qquad i\in\mathbb{N}.$$

Definition

Define a subset $\mathscr{R}$ of $\mathbb{N}\times\mathbb{N}$ by

$$\Big((k,l)\in\mathscr{R}\Big)\iff\Big(s_{k,l}\neq 0\Big).$$

Theorem

The subset $\mathscr{R}\subset\mathbb{N}\times\mathbb{N}$ is an equivalence relation.

- We interpret the equivalence as being in the same orbit of the action of $\mathbb{G}$.
- In case of the action of $\mathbb{G}$ on $\operatorname{Irr}\mathbb{K}$ this is the action on irreps by automorphisms induced on $\mathbb{K}$ by inner automorphisms of $\mathbb{G}$.

Proposition

For any $i \in \mathbb{N}$ let $[i]$ be the class of i w.r.t. the relation $\mathscr{R}$. The element

$$x = \sum_{j \in [i]} e_j$$

of $\ell^\infty(\mathbb{N})$ is invariant:

$$\alpha(x) = x \otimes \mathbb{1}.$$

Moreover α restricts to an action of $\mathbb{G}$ on $x\,\ell^\infty(\mathbb{N}) = \ell^\infty([i])$.

Corollary

If α is ergodic, then $\mathscr{R}$ is a total relation.

Theorem

There does not exist an ergodic action of a compact quantum group on $\ell^\infty(\mathbb{N})$.

Proof.

We have an invariant faithful state ϱ on $\ell^\infty(\mathbb{N})$. We have

$$\sum_{i=1}^{\infty} \varrho(e_i) s_{i,j} = \varrho(e_j)\mathbb{1}, \qquad \text{(invariance)}$$

$$\sum_{i=1}^{\infty} \varrho(e_i) = 1 \qquad (\varrho \text{ is a state})$$

Take ξ a unit vector in the range of $s_{j,j}$ and apply $\langle\xi|\cdot\xi\rangle$ to both sides of the first equation

$$\varrho(e_j) + \sum_{i\neq j} \varrho(e_i)\left\langle \xi \middle| s_{i,j}\xi \right\rangle = \varrho(e_j).$$

Since all $\varrho(e_i)$ are non zero, we find that for $i \neq j$ the range of $s_{i,j}$ is orthogonal to the range of $s_{j,j}$.

PROOF.

Fix $k \neq j$ and let η be a unit vector in the range of $s_{k,j}$. Apply $\langle\eta|\cdot\eta\rangle$ to both sides of

$$\varrho(e_j)\mathbb{1} = \sum_{i=1}^{\infty} \varrho(e_i)s_{i,j}$$

to get

$$\varrho(e_j) = \varrho(e_j)\left\langle\eta\middle|s_{j,j}\eta\right\rangle + \varrho(e_k)\left\langle\eta\middle|s_{k,j}\eta\right\rangle + \sum_{k\neq i\neq j}^{\infty} \varrho(e_i)\left\langle\eta\middle|s_{i,j}\eta\right\rangle.$$

The first term on the right is 0, the second is $\varrho(e_k)$ and the third is ≥ 0, so $\varrho(e_j) \leq \varrho(e_k)$. By symmetry all $\varrho(e_i)$'s are equal which contradicts $\sum\limits_{i=1}^{\infty} \varrho(e_i) = 1$. $\square$

Corollary

- *The orbits of α are finite,*
- *the counting measure on $\mathbb{N}$ is invariant for α, in particular*

$$\sum_{i=1}^{\infty} s_{i,j} = \sum_{i\in[j]} s_{i,j} = \mathbb{1}, \qquad j \in \mathbb{N}.$$

Theorem

Let $\mathbb{K}$ be a normal closed quantum subgroup of the compact quantum group $\mathbb{G}$. Let π be an irreducible representation of $\mathbb{G}$. Then there exists an irreducible representation σ of $\mathbb{K}$ such that

1. *for all irreducible representations τ of $\mathbb{K}$ we have $\pi(p_\tau) \neq 0$ if and only if $\tau \in [\sigma]$,*
2. *we have $\pi\left(\sum_{\rho\in[\sigma]} p_\rho\right) = \mathbb{1}$.*

- This becomes Clifford's theorem in the classical case.
- Classically degrees of all irreps in one orbit are equal.

THEOREM

Assume $\mathbb{G}$ is of Kac type. Then if σ_1 and σ_2 are irreducible representations of $\mathbb{K}$ in the same orbit under the action of $\mathbb{G}$ then for any irreducible representation π of $\mathbb{G}$ we have

$$\dim \pi(p_{\sigma_1}) = \dim \pi(p_{\sigma_2}).$$

- This means that the dimension of isotypical components corresponding to σ_1 and σ_2 are the same in the restriction of π to $\mathbb{K}$.

Theorem

Let $\sigma \in \operatorname{Irr}\mathbb{K}$. Then the sum

$$\sum_{\rho\in[\sigma]} p_\rho$$

is the central support of p_σ in $\ell^\infty(\widehat{\mathbb{G}})$.

- Note that all projections above are central in $\ell^\infty(\widehat{\mathbb{K}})$.

- Let $\mathbb{\Gamma}$ be a discrete quantum group and $\mathbb{\Lambda}$ its closed quantum subgroup.
- Then $\mathbb{\Lambda}$ is discrete and $\operatorname{Irr}\widehat{\mathbb{\Lambda}} \subset \operatorname{Irr}\widehat{\mathbb{\Gamma}}$.
- R. Vergnioux and later R. Vergnioux & C. Voigt used the following relation on $\operatorname{Irr}\widehat{\mathbb{\Gamma}}$:

$$\Bigl(\alpha \approx \beta\Bigr) \Longleftrightarrow \Bigl(\exists\,\gamma \in \operatorname{Irr}\widehat{\mathbb{\Lambda}} \text{ s.t. } \beta \subset \alpha \mathbin{\bigcirc\!\!\!\!\top} \gamma\Bigr)$$

- $\approx$ is an equivalence relation.
- For a class A of this relation we write

$$p_A = \sum_{\alpha \in A} p_\alpha.$$

- The set of such p_A's over all distinct classes is a collection of pairwise orthogonal central projections in $\ell^\infty(\mathbb{\Gamma})$ summing up to $\mathbb{1}$.

- Consider a particular case:
  - $\mathbb{\Gamma} = \widehat{\mathbb{G}}$,
  - $\mathbb{\Lambda} = \widehat{\mathbb{G}/\mathbb{K}}$.

THEOREM

Let A be a class of the equivalence relation $\approx$ on $\operatorname{Irr}\mathbb{G}$. Then there exists an irreducible representation σ of $\mathbb{K}$ such that

$$p_A = \sum_{\rho \in [\sigma]} p_\rho.$$

In particular

$$z(p_\sigma) = \sum_{\alpha \in A} p_\alpha$$

is the expression of the central projection $z(p_\sigma)$ as a sum of orthogonal minimal central projections.

Thank you for your attention.