The quantum disk is not a quantum group

Piotr M. Sołtan

joint work with Jacek Krajczok

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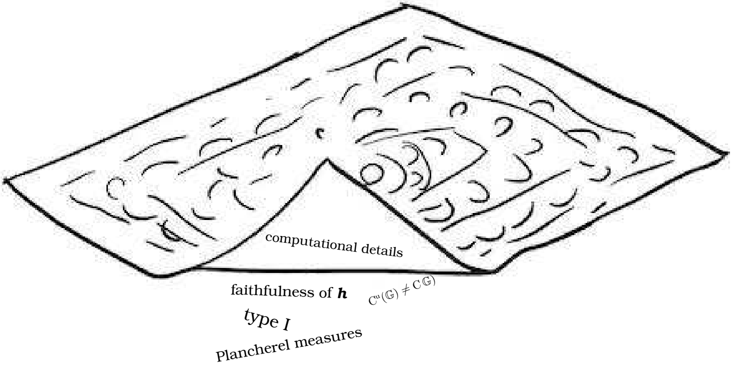
computational details
faithfulness of $\boldsymbol{h}$
$\mathrm{C^u}(\mathbb{G}) \neq \mathrm{C}(\mathbb{G})$
type I
Plancherel measures

DEFINITION (S.L. WORONOWICZ)

A **compact quantum group** is a compact quantum space $\mathbb{G}$ such that there exists

$$\Delta\colon \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$$

such that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and

$$\begin{gathered}\bigl\{\Delta(a)(\mathbb{1} \otimes b) \,\big|\, a, b \in \mathrm{C}(\mathbb{G})\bigr\},\\ \bigl\{(a \otimes \mathbb{1})\Delta(b) \,\big|\, a, b \in \mathrm{C}(\mathbb{G})\bigr\}\end{gathered}$$

are linearly dense in $\mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$.

Example (S.L. Woronowicz)

The quantum $\mathrm{SU}(2)$

- Choose $q \in [-1, 1] \setminus \{0\}$.
- Let $\mathrm{C}(\mathbb{G})$ be the C*-algebra generated by α and γ such that

$$\begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix}$$

  is unitary.
- There a unique exists $\Delta\colon \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$ such that

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

- With this Δ, the quantum space $\mathbb{G}$ is a compact quantum group: $\mathbb{G} = \mathrm{SU}_q(2)$.

Some interesting facts

- The quantum tori are not compact quantum groups.
- Some quantum 2-spheres admit no compact quantum group structure:
  - the standard Podleś sphere,
  - the Bratteli-Elliott-Evans-Kishimoto quantum spheres,
  - the Natsume-Olsen quantum spheres,
  - the generic Podleś spheres (work in progress).
- A nuclear C^*-algebra admitting a faithful family of traces and a character cannot be $C(\mathbb{G})$ for a compact quantum group.

- The **quantum disk** is the quantum space $\mathbb{U}$ such that $\mathrm{C}(\mathbb{U})$ is isomorphic to the Toeplitz algebra $\mathcal{T}$.
- First mention: A. Sheu 1991. (As far as I know.)
- In-depth analysis: Klimek-Leśniewski 1992.
- We have the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow \mathrm{C}(\mathbb{T}) \longrightarrow 0 \,.$$

- Assume there is a compact quantum group $\mathbb{G}$ such that we have an isomorphism

$$\pi_\bullet\colon\ \mathrm{C}(\mathbb{G}) \longrightarrow \mathcal{T}.$$

- Goal: arrive at a contradiction.

Definition

Let $\mathbb{G}$ be a compact quantum group.

- The **Haar measure** of $\mathbb{G}$ is a state $\boldsymbol{h}$ on $\mathrm{C}(\mathbb{G})$ such that

$$(\mathrm{id}\otimes\boldsymbol{h})\Delta(a)=\boldsymbol{h}(a)\mathbb{1}=(\boldsymbol{h}\otimes\mathrm{id})\Delta(a),\qquad a\in\mathrm{C}(\mathbb{G}).$$

- A compact quantum group is of **Kac type** is $\boldsymbol{h}$ is a trace.

Theorem (S.L. Woronowicz)

There exists a one-parameter family $\{f_{\mathrm{i}t}\}_{t\in\mathbb{R}}$ of characters of $\mathrm{C}(\mathbb{G})$ such that the modular group of $\boldsymbol{h}$ is given by

$$\sigma^{\boldsymbol{h}}_t(a)=(f_{\mathrm{i}t}\otimes\mathrm{id}\otimes f_{\mathrm{i}t})(\Delta\otimes\mathrm{id})\Delta(a),\qquad a\in\mathrm{C}(\mathbb{G}).$$

Moreover, $\mathbb{G}$ is of Kac type iff the family $\{f_{\mathrm{i}t}\}_{t\in\mathbb{R}}$ is trivial.

Proposition

$\mathbb{G}$ is not of Kac type and Woronowicz characters coincide with evaluations at points of $\mathbb{T}$ though the symbol map

$$\mathrm{C}(\mathbb{G}) \xrightarrow{\pi_\bullet} \mathcal{T} \longrightarrow\!\!\!\!\to \mathrm{C}(\mathbb{T}).$$

- The formula

$$\tau_t(a) = (f_{\mathrm{i}t} \otimes \mathrm{id} \otimes f_{-\mathrm{i}t})(\Delta \otimes \mathrm{id})\Delta(a), \qquad a \in \mathrm{C}(\mathbb{G})$$

  defines a one-parameter group of automorphisms of $\mathrm{C}(\mathbb{G})$.
- We call this group the **scaling group**.
- A compact quantum group is Kac type iff its scaling group is trivial.

PROPOSITION

Let $U \in \mathrm{M}_n(\mathbb{C}) \otimes \mathrm{C}(\mathbb{G})$ be an irreducible unitary representation of the compact quantum group $\mathbb{G}$:

$$U = \begin{bmatrix} U_{1,1} & \cdots & U_{1,n} \\ \vdots & \ddots & \vdots \\ U_{n,1} & \cdots & U_{n,n} \end{bmatrix}.$$

Then $\pi_\bullet(U_{i,j})$ is

- *compact if $i \neq j$,*
- *Fredholm if $i = j$.*

- The compact quantum group $\mathbb{G}$ has its **Pontriagin dual** $\widehat{\mathbb{G}}$.
- $\widehat{\mathbb{G}}$ is a **discrete quantum group**. (Rightfully so.)
- We have $\mathrm{C}(\mathbb{G}) \cong \mathrm{C}^*(\widehat{\mathbb{G}})$.
- $\mathrm{C}(\mathbb{G})$ is a type I C^*-algebra, so $\widehat{\mathbb{G}}$ is a **type** I **quantum group**.
- Let $\mathsf{L}_\infty(\mathbb{G})$ be the strong closure of $\mathrm{C}(\mathbb{G})$ in the GNS representation for the Haar measure $\boldsymbol{h}$.

THEOREM (DESMEDT, KRAJCZOK)

$\mathsf{L}_\infty(\mathbb{G})$ *is unitarily equivalent to*

$$\int\limits_{\operatorname{spec}\mathcal{T}}^{\oplus} \bigl(\mathrm{B}(\mathsf{H}_x) \otimes \mathbb{1}_{\overline{\mathsf{H}_x}}\bigr)\, d\mu(x)$$

for a certain standard measure on $\operatorname{spec}\mathcal{T} = \{\bullet\} \cup \mathbb{T}$.

- $\mathsf{L}_\infty(\mathbb{G}) \cong \bigl(\mathrm{B}(\mathsf{H}_\bullet) \otimes \mathbb{1}_{\overline{\mathsf{H}_\bullet}}\bigr) \oplus \int\limits_{\mathbb{T}}^{\oplus} \bigl(\mathrm{B}(\mathsf{H}_z) \otimes \mathbb{1}_{\overline{\mathsf{H}_z}}\bigr)\, d\mu(z)$.
- μ is called the **Plancherel measure**.

- The theory of type I quantum groups provides the formula for the Haar measure

$$\boldsymbol{h}(a)=\int\limits_{\operatorname{spec}\mathrm{C}(\mathbb{G})}\operatorname{Tr}\bigl(\pi_x(a)D_x^{-2}\bigr)\,d\mu(x),$$

  for appropriate
  - field of representations $x\mapsto\pi_x$,
  - field of positive self-adjoint non-singular operators $x\mapsto D_x$.
- In particular

$$\boldsymbol{h}(1)=\operatorname{Tr}(D_\bullet^{-2})+\int\limits_{\mathbb{T}}\operatorname{Tr}(D_z^{-2})\,d\mu(z),$$

  so $D_\bullet^{-1}$ is a Hilbert-Schmidt operator.

PROPOSITION

Fro $a \in \mathcal{T}$ and $t \in \mathbb{R}$ we have

$$\pi_\bullet\Big(\sigma_t^{\boldsymbol{h}}\big(\pi_\bullet^{-1}(a)\big)\Big) = D_\bullet^{-2\mathrm{i}t} a D_\bullet^{2\mathrm{i}t},$$
$$\pi_\bullet\Big(\tau_t\big(\pi_\bullet^{-1}(a)\big)\Big) = B^{-\mathrm{i}t} a B^{\mathrm{i}t}$$

for a certain positive self-adjoint non-singular operator B strongly commuting with $D_\bullet$.

COROLLARY

B preserves the decomposition

$$\mathsf{H}_\bullet = \bigoplus_{q \in \mathrm{Sp}\, D_\bullet^{-1}} \mathsf{H}_\bullet(D_\bullet^{-1} = q).$$

- Based on the fact that $\mathbb{G}$ is not Kac type we make an informed choice of the irrep U.
- Now we consider the Fredholm operator $\pi_\bullet(U_{i,i})$ for appropriate i.
- We show that $\pi_\bullet(U_{i,i})$ shifts each subspace

$$\mathsf{H}_\bullet(D_\bullet^{-1} = q)$$

  into

$$\mathsf{H}_\bullet(D_\bullet^{-1} = q\rho)$$

  for some $\rho > 1$ (connected with the choices we made above).
- Then $\pi_\bullet(U_{i,i})$ must be injective on $\mathsf{H}_\bullet(D_\bullet^{-1} = q)$ for $q < q_0$.

- We let Λ_q denote the spectrum of B restricted to the subspace $\mathsf{H}_\bullet(D_\bullet^{-1} = q)$. $\dim \mathsf{H}_\bullet(D_\bullet^{-1} = q) < +\infty$
- Due to injectivity of $\pi_\bullet(U_{i,i})$ on almost all of these subspaces, the set

$$\bigcup_{q \in \operatorname{Sp} D_\bullet^{-1}} \Lambda_q$$

  is finite.
- One can conclude from this that B and B^{-1} are *bounded*.
- By some arcane results of the theory of compact quantum groups this implies that $\tau_t = \mathrm{id}$ for all $t \in \mathbb{R}$.
- This means that G is of *Kac type* — a contradiction 😊

Thank you for listening!