Quantum Families of Maps

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THEOREM

The assignment to a locally compact topological space X the C^**-algebra* $\mathrm{C}_0(X)$ *defines an anti-equivalence of categories between*

- *the category of commutative* C^**-algebras with morphisms of* C^**-algebras*

and

- *the category of locally compact topological spaces with continuous maps.*

- A locally compact space is by definition Hausdorff.
- The "inverse" functor is defined as the assignment to a commutative C^*-algebra A its spectrum $\widehat{\mathsf{A}}$.
- A morphism of C^*-algebras from A to B is a $*$-homomorphism $\Phi:\mathsf{A}\to\mathrm{M}(\mathsf{B})$ such that $\overline{\Phi(\mathsf{A})\mathsf{B}}=\mathsf{B}$.

L.C. Top. Sp.	**Commutative C^*-algs.**
X	$\mathrm{C}_0(X)$
$\varphi : X \to Y$	$\Phi \in \mathrm{Mor}\big(\mathrm{C}_0(Y)\,\mathrm{C}_0(X)\big)$
X – compact	$\mathrm{C}_0(X)$ – unital
X – finite	$\mathrm{C}_0(X)$ – finite-dimensional
X – metrizable	$\mathrm{C}_0(X)$ – separable
probab. measure on X	state on $\mathrm{C}_0(X)$
$X \times Y$	$\mathrm{C}_0(X) \otimes \mathrm{C}_0(Y)$

- Note: $\mathrm{M}\big(\mathrm{C}_0(X)\big) = \mathrm{C}_\mathrm{b}(X)$.

Definition

A **Quantum space** is an object of the category dual to the category of C*-algebras.

- A theorem about quantum spaces is nothing else than a theorem about C*-algebras.
- A quantum space $\mathcal{X}$ is called **compact** if the corresponding C*-algebra $C_0(\mathcal{X})$ is unital (in this case we write $C(\mathcal{X})$).
- Similarly, **finite** quantum spaces correspond to finite-dimensional C*-algebras.
- Classical (ordinary) locally compact spaces are particular examples of quantum spaces.

THEOREM (JAMES R. JACKSON, 1952)

Let X, Y and Z be topological spaces such that X is Hausdorff and Z is locally compact. Then the assignment to any $\psi \in \mathrm{C}(X \times Z, Y)$ of the map

$$Z \ni z \longmapsto \psi(\,\cdot\,, z) \in \mathrm{C}(X, Y)$$

is a homeomorphism of $\mathrm{C}(X \times Z, Y)$ onto $\mathrm{C}\big(Z, \mathrm{C}(X, Y)\big)$ with all three spaces of maps topologized by their respective compact-open topologies.

- Assume that X, Y and Z are locally compact. Then a continuous family of continuous maps from X to Y indexed by Z, i.e. a continuous map from Z to $\mathrm{C}(X, Y)$ is the same thing as an element of

$$\mathrm{Mor}\big(\mathrm{C}_0(Y), \mathrm{C}_0(X) \otimes \mathrm{C}_0(Z)\big).$$

DEFINITION

Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be quantum spaces. A **quantum family of maps** from $\mathcal{X}$ to $\mathcal{Y}$ indexed by $\mathcal{Z}$ is an element

$$\Phi \in \mathrm{Mor}\big(\mathrm{C}_0(\mathcal{Y}), \mathrm{C}_0(\mathcal{X}) \otimes \mathrm{C}_0(\mathcal{Z})\big).$$

- A quantum family of maps is a very general object.
- Consequently interesting quantum families of maps must have additional features.
- How about a quantum version of the space $\mathrm{C}(X, Y)$ of all continuous maps from X to Y?

Definition

Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be quantum spaces and let $\Phi \in \operatorname{Mor}\big(\mathrm{C}_0(\mathcal{Y}), \mathrm{C}_0(\mathcal{X}) \otimes \mathrm{C}_0(\mathcal{Z})\big)$ be a quantum family of maps. We say that

- $\mathcal{Z}$ is the **quantum space of all maps** from $\mathcal{X}$ to $\mathcal{Y}$

and

- Φ is the **quantum family of all maps** from $\mathcal{X}$ to $\mathcal{Y}$

if for any quantum space $\mathcal{Z}'$ and any quantum family $\Psi \in \operatorname{Mor}\big(\mathrm{C}_0(\mathcal{Y}), \mathrm{C}_0(\mathcal{X}) \otimes \mathrm{C}_0(\mathcal{Z}')\big)$ there exists a unique $\Lambda \in \operatorname{Mor}\big(\mathrm{C}_0(\mathcal{Z}), \mathrm{C}_0(\mathcal{Z}')\big)$ such that

$$\begin{array}{ccc} \mathrm{C}_0(\mathcal{Y}) & \xrightarrow{\quad\Phi\quad} & \mathrm{C}_0(\mathcal{X}) \otimes \mathrm{C}_0(\mathcal{Z}) \\ \Big\| & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\Lambda} \\ \mathrm{C}_0(\mathcal{Y}) & \xrightarrow{\quad\Psi\quad} & \mathrm{C}_0(\mathcal{X}) \otimes \mathrm{C}_0(\mathcal{Z}') \end{array}$$

- Let X, Y and Z be locally compact spaces. For the quantum space of all maps from X to Y to exist it is necessary that $\mathrm{C}(X, Y)$ be locally compact in the compact-open topology.
- This will certainly be the case when X is finite and Y is compact.

THEOREM (S.L. WORONOWICZ 1979, P.S. 2009)

Let $\mathcal{X}$ be a finite quantum space and let $\mathcal{Y}$ be a compact quantum space such that $\mathrm{C}(\mathcal{Y})$ is finitely generated. Then the quantum space of all maps from $\mathcal{X}$ to $\mathcal{Y}$ exists and it is compact.

- Let $\mathcal{Z}$ be the quantum space of all maps $\mathcal{X} \to \mathcal{Y}$.
- The universal family

$$\boldsymbol{\Phi} \in \mathrm{Mor}\big(\mathrm{C}(\mathcal{Y}), \mathrm{C}(\mathcal{X}) \otimes \mathrm{C}(\mathcal{Z})\big)$$

corresponds to the evaluation mapping

$$X \times \mathrm{C}(X, Y) \ni (x, \psi) \longmapsto \psi(x) \in Y$$

for classical spaces X and Y.

EXAMPLE

- Let X be the classical two-point space.
- The classical space of all maps $X \to X$ is a four-point space.
- The quantum space $\mathcal{Z}$ of all maps $X \to X$ exists and

$$\begin{aligned}\mathrm{C}(\mathcal{Z}) &\cong \mathbb{C}^2 * \mathbb{C}^2 \cong \mathrm{C}^*(\mathbb{Z}_2 * \mathbb{Z}_2) \cong \mathrm{C}(\mathbb{T}) \rtimes \mathbb{Z}_2 \\ &\cong \left\{ f \in \mathrm{C}\big([0,1], M_2\big) \,\middle|\, f(0), f(1) \text{ are diagonal} \right\}\end{aligned}$$

and $\boldsymbol{\Phi} : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathrm{C}(\mathcal{Z})$ maps $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \otimes p + \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \otimes q$, where

$$p(t) = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \qquad q(t) = \tfrac{1}{2}\begin{bmatrix}1 - \cos \pi t & \mathrm{i} \sin \pi t \\ -\mathrm{i} \sin \pi t & 1 + \cos \pi t\end{bmatrix}.$$

Definition

Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{P}_1$ and $\mathcal{P}_2$ be quantum spaces and let

$$\Psi_{12} \in \mathrm{Mor}\big(\mathrm{C}(\mathcal{X}_2), \mathrm{C}(\mathcal{X}_1) \otimes \mathrm{C}(\mathcal{P}_1)\big),$$
$$\Psi_{23} \in \mathrm{Mor}\big(\mathrm{C}(\mathcal{X}_3), \mathrm{C}(\mathcal{X}_2) \otimes \mathrm{C}(\mathcal{P}_2)\big)$$

be quantum families of maps. The **composition** of Ψ_{32} and Ψ_{21} is

$$\Psi_{12} \triangle \Psi_{23} = (\Psi_{12} \otimes \mathrm{id}) \circ \Psi_{23} \in \mathrm{Mor}\big(\mathrm{C}(\mathcal{X}_3), \mathrm{C}(\mathcal{X}_1) \otimes (\mathrm{C}(\mathcal{P}_2) \otimes \mathrm{C}(\mathcal{P}_2))\big).$$

- The composition of classical families (with classical parameter spaces) is exactly the family of all compositions of elements of both families.
- Composition is associative:

$$\big(\Psi_{12} \triangle \Psi_{23}\big) \triangle \Psi_{34} = \Psi_{12} \triangle \big(\Psi_{23} \triangle \Psi_{34}\big).$$

THEOREM

Let $\mathcal{X}$ be a finite quantum space and let $\mathcal{Z}$ be the quantum space of all maps $\mathcal{X} \to \mathcal{X}$ with universal family

$$\boldsymbol{\Phi} \in \mathrm{Mor}\big(\mathrm{C}(\mathcal{X}), \mathrm{C}(\mathcal{X}) \otimes \mathrm{C}(\mathcal{Z})\big).$$

Then there exists a unique

$$\Delta \in \mathrm{Mor}\big(\mathrm{C}(\mathcal{Z}), \mathrm{C}(\mathcal{Z}) \otimes \mathrm{C}(\mathcal{Z})\big)$$

such that $(\mathrm{id} \otimes \Delta) \circ \boldsymbol{\Phi} = \boldsymbol{\Phi} \vartriangle \boldsymbol{\Phi}$.
This defines on $\mathcal{Z}$ a structure of a compact quantum semigroup.

- We have $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$.
- $\mathcal{Z}$ is not a compact quantum group (unless $\dim \mathrm{C}(\mathcal{X}) = 1$).
- For $\mathcal{X}$ – a two-point space

$$\Delta(p) = p \otimes (\mathbb{1} - p) + \mathbb{1} \otimes \mathbb{1} + (\mathbb{1} - p) \otimes p$$
$$\Delta(q) = q \otimes (\mathbb{1} - q) + \mathbb{1} \otimes \mathbb{1} + (\mathbb{1} - q) \otimes q.$$

- $\mathcal{X}$ – finite quantum space, ω – state on $\mathrm{C}(\mathcal{X})$.
- A quantum family of maps

$$\Psi \in \mathrm{Mor}\big(\mathrm{C}(\mathcal{X}), \mathrm{C}(\mathcal{X}) \otimes \mathrm{C}(\mathcal{M})\big)$$

  preserves ω (or ω is **invariant** for Ψ) if

$$(\omega \otimes \mathrm{id})\Psi(a) = \omega(a)\mathbb{1}, \qquad a \in \mathrm{C}(\mathcal{X}).$$

- The **quantum family of all maps preserving** ω is a quantum family

$$\boldsymbol{\Phi}_\omega \in \mathrm{Mor}\big(\mathrm{C}(\mathcal{X}), \mathrm{C}(\mathcal{X}) \otimes \mathrm{C}(\mathcal{Z}_\omega)\big)$$

  such that for any quantum family Ψ preserving ω (as above) there exists a unique $\Gamma : \mathrm{C}(\mathcal{Z}_\omega) \to \mathrm{C}(\mathcal{M})$ such that

$$\begin{array}{ccc} \mathrm{C}_0(\mathcal{X}) & \xrightarrow{\boldsymbol{\Phi}_\omega} & \mathrm{C}_0(\mathcal{X}) \otimes \mathrm{C}_0(\mathcal{Z}_\omega) \\ \Big\| & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\Gamma} \\ \mathrm{C}_0(\mathcal{X}) & \xrightarrow{\Psi} & \mathrm{C}_0(\mathcal{X}) \otimes \mathrm{C}_0(\mathcal{M}) \end{array}$$

Example

- Put $X_n = \{1, \ldots, n\}$.
- Let ω be the uniform probability measure on X_n.
- Then
  - the quantum space $\mathcal{Z}_\omega$ of all maps $X_n \to X_n$ preserving ω is the quantum permutation group S_n^+,
  - the universal family

$$\mathbf{\Phi}_\omega \in \mathrm{Mor}\big(\mathrm{C}(X_n), \mathrm{C}(X_n) \otimes \mathrm{C}(\mathcal{Z}_\omega)\big)$$

is the action of S_n^+ on X_n.

DEFINITION

Let $\mathcal{X}$ be a finite quantum space and let

$$\Psi_1 \in \mathrm{Mor}\big(\mathrm{C}(\mathcal{X}), \mathrm{C}(\mathcal{X}) \otimes \mathrm{C}(\mathcal{P}_1)\big)$$
$$\Psi_2 \in \mathrm{Mor}\big(\mathrm{C}(\mathcal{X}), \mathrm{C}(\mathcal{X}) \otimes \mathrm{C}(\mathcal{P}_2)\big)$$

be a quantum families of maps.
We say that Ψ_1 and Ψ_2 **commute** if

$$\Psi_1 \triangle \Psi_2 = (\mathrm{id} \otimes \chi) \circ (\Psi_2 \triangle \Psi_1),$$

where $\chi : \mathrm{C}(\mathcal{P}_2) \otimes \mathrm{C}(\mathcal{P}_1) \to \mathrm{C}(\mathcal{P}_1) \otimes \mathrm{C}(\mathcal{P}_2)$ is the flip.

THEOREM

For any quantum family $\Psi \in \mathrm{Mor}\big(\mathrm{C}(\mathcal{X}), \mathrm{C}(\mathcal{X}) \otimes \mathrm{C}(\mathcal{P})\big)$ *there exists a universal quantum family* $\boldsymbol{\Phi}_\Psi \in \mathrm{Mor}\big(\mathrm{C}(\mathcal{X}), \mathrm{C}(\mathcal{X}) \otimes \mathrm{C}(\mathcal{Z}_\Psi)\big)$ *commuting with* Ψ.

- Classically if X is a topological space and S is a topological semigroup then $\mathrm{C}(X, S)$ has a structure of a topological semigroup.
- For simplicity let $X = \{1, \ldots, n\}$.
- If S is a compact semigroup then $\mathrm{C}(X, S) = S^X$ is a compact semigroup and comultiplication on $\mathrm{C}(S^X)$ is the unique Δ_{S^X} such that

$$
\begin{array}{ccc}
\mathrm{C}(S) & \xrightarrow{\quad\Phi\quad} & \mathrm{C}(X) \otimes \mathrm{C}(S^X) \\
\Big\downarrow{\scriptstyle \Delta_S} & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\Delta_{S^X}} \\
\mathrm{C}(S) \otimes \mathrm{C}(S) & & \mathrm{C}(X) \otimes \mathrm{C}(S^X) \otimes \mathrm{C}(S^X) \\
\Big\downarrow{\scriptstyle \Phi\otimes\Phi} & & \Big\uparrow{\scriptstyle \mu\otimes\mathrm{id}\otimes\mathrm{id}} \\
\mathrm{C}(X) \otimes \mathrm{C}(S^X) \otimes \mathrm{C}(X) \otimes \mathrm{C}(S^X) & \xrightarrow[\mathrm{id}\otimes\chi\otimes\mathrm{id}]{} & \mathrm{C}(X) \otimes \mathrm{C}(X) \otimes \mathrm{C}(S^X) \otimes \mathrm{C}(S^X)
\end{array}
$$

$\Phi : \mathrm{C}(S) \to \mathrm{C}(X) \otimes \mathrm{C}(S^X)$ – evaluation,
$\mu : \mathrm{C}(X) \otimes \mathrm{C}(X) \to \mathrm{C}(X)$ – multiplication.

Theorem (M.M. Sadr, P.S.)

Let $\mathcal{S}$ be a compact quantum semigroup and let $\widetilde{\mathcal{S}}$ be the quantum space of all maps $X_n \to \mathcal{S}$. Then there exists a unique $\widetilde{\Delta}$ such that

$$\begin{array}{ccc}
\mathrm{C}(\mathcal{S}) & \xrightarrow{\quad\Phi\quad} & \mathrm{C}(X_n)\otimes\mathrm{C}(\widetilde{\mathcal{S}}) \\
\Big\downarrow{\scriptstyle \Delta_{\mathcal{S}}} & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\widetilde{\Delta}} \\
\mathrm{C}(\mathcal{S})\otimes\mathrm{C}(\mathcal{S}) & & \mathrm{C}(X_n)\otimes\mathrm{C}(\widetilde{\mathcal{S}})\otimes\mathrm{C}(\widetilde{\mathcal{S}}) \\
\Big\downarrow{\scriptstyle \Phi\otimes\Phi} & & \Big\uparrow{\scriptstyle \mu\otimes\mathrm{id}\otimes\mathrm{id}} \\
\mathrm{C}(X_n)\otimes\mathrm{C}(\widetilde{\mathcal{S}})\otimes\mathrm{C}(X_n)\otimes\mathrm{C}(\widetilde{\mathcal{S}}) & \xrightarrow[\mathrm{id}\otimes\chi\otimes\mathrm{id}]{} & \mathrm{C}(X_n)\otimes\mathrm{C}(X_n)\otimes\mathrm{C}(\widetilde{\mathcal{S}})\otimes\mathrm{C}(\widetilde{\mathcal{S}})
\end{array}$$

If $\mathcal{S}$ is a compact quantum group then so is $\widetilde{\mathcal{S}}$. The C^*-algebra $\mathrm{C}(\widetilde{\mathcal{S}})$ is isomorphic to $\underbrace{\mathrm{C}(\mathcal{S}) * \cdots * \mathrm{C}(\mathcal{S})}_{n}$.

- For a non-classical finite quantum space $\mathcal{X}$ the diagram defining comultiplication on the quantum space of all maps $\mathcal{X} \to \mathcal{S}$ makes no sense because μ is not a homomorphism.
- Nevertheless the quantum space of all maps from $\mathcal{X}$ to $\mathcal{S}$ exists…

Example

- Consider $\mathcal{X}$ such that $\mathrm{C}(\mathcal{X}) = M_2$ and take $\mathcal{S} = \mathbb{Z}_2$ (classical finite group).
- Let $\widetilde{\mathcal{S}}$ be the quantum space of all maps $\mathcal{X} \to \mathcal{S}$.
- Then $\mathrm{C}(\widetilde{\mathcal{S}})$ is the universal unital C*-algebra generated by three elements p, q and z with relations

$$
\begin{aligned}
p &= p^*, & p &= p^2 + z^*z, & zp &= (\mathbb{1} - q)z,\\
q &= q^*, & q &= q^2 + zz^*.
\end{aligned}
$$

- $\mathrm{C}(\widetilde{\mathcal{S}})$ does not admit **any** compact quantum group structure.

Thank you for your attention.