When a quantum space is not a group?

Piotr M. Sołtan

Department of Mathematical Methods in Physics
Faculty of Physics, Warsaw University

July 23 2009 (Banach Algebras 2009)

Plan

- Any topological space X can be given a structure of an associative topological semigroup (e.g. $x \cdot y = y$).
- Can $[0,1]$ be given a structure of a topological group?
- The same goes for any manifold with boundary.
- A related question: let

$$X = \left\{ \begin{bmatrix} s \\ t \\ r \end{bmatrix} \in \mathbb{C}^3 \;\middle|\; \begin{array}{c} st = -r^2, \\ |s| + |t| = 1 \end{array} \right\}$$

and define multiplication on X by

$$\begin{bmatrix} s \\ t \\ r \end{bmatrix} \cdot \begin{bmatrix} s' \\ t' \\ r' \end{bmatrix} = \begin{bmatrix} 2(r\bar{t} - s\bar{r})r' + ss' + \bar{t}t' \\ 2(t\bar{r} - r\bar{s})r' + ts' + \bar{s}t' \\ \left(|s|^2 - |t|^2\right)r' + rs' + \bar{r}t' \end{bmatrix}.$$

Is X a topological group?

- We will investigate problems of existence of group structure on compact quantum spaces.
- A **compact quantum space** is an object of the category dual to the category of unital C^*-algebras.
- A **compact quantum semigroup** is a pair
  - (A, Δ)
    - A — unital C^*-algebra
    - Δ — unital $*$-homomorphism $A \to A \otimes A$
  - $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$.
- Example: $A = \mathsf{C}(S)$ $\quad$ (S — compact semigroup),

$$\Delta(f) \in A \otimes A = \mathsf{C}(S \times S), \qquad \Delta(f)(s,t) = f(st).$$

- A **compact quantum group** is a compact quantum semigroup (A, Δ) such that

$$\operatorname{span}\left\{(a \otimes \mathbf{1})\Delta(b) \,\middle|\, a, b \in A\right\} \subset_{\text{dense}} A \otimes A,$$
$$\operatorname{span}\left\{\Delta(a)(\mathbf{1} \otimes b) \,\middle|\, a, b \in A\right\} \subset_{\text{dense}} A \otimes A.$$

- In case $A = \mathrm{C}(S)$ density conditions correspond to

$$\left(s \cdot t = s \cdot t'\right) \Rightarrow \left(t = t'\right),$$
$$\left(s \cdot t = s' \cdot t\right) \Rightarrow \left(s = s'\right).$$

- Example:
  - $A = \mathrm{C}^*(\Gamma)$ (Γ — discrete group),
  - $\Delta(\gamma) = \gamma \otimes \gamma$ ($\gamma \in \Gamma$).

Haar measure

Theorem (S.L. Woronowicz)

Let (A, Δ) be a compact quantum group. Then there exists a unique state h on A such that

$$(\mathrm{id} \otimes h)\Delta(a) = (h \otimes \mathrm{id})\Delta(a) = h(a)\mathbf{1}$$

for all $a \in A$.

- For $A = \mathrm{C}(G)$ (G — compact group)

$$h(f) = \int_G f(t)\, dt.$$

- For $A = \mathrm{C}^*(\Gamma)$ (Γ — discrete group)

$$h(\gamma) = \delta_{\gamma,e}$$

for $\gamma \in \Gamma$. (This might not be faithful.)

Reduced quantum group

- (A, Δ) — compact quantum group, h — it's Haar measure.
- Let $J = \left\{ a \in A \middle| h(a^* a) = 0 \right\}$, $A_r = A/J$, $\lambda : A \twoheadrightarrow A_r$.
- There is a unique $\Delta_r : A_r \to A_r \otimes A_r$ such that

$$\begin{array}{ccc} A & \xrightarrow{\quad\Delta\quad} & A \otimes A \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\lambda\otimes\lambda} \\ A_r & \xrightarrow[\quad\Delta_r\quad]{} & A_r \otimes A_r \end{array}$$

- (A_r, Δ_r) is a compact quantum group — **reduced** (A, Δ).
- For $A = \mathrm{C}^*(\Gamma)$ we have $A_r = \mathrm{C}^*_r(\Gamma)$.

Hopf algebra

- (A, Δ) — compact quantum group.
- There exists a unique dense unital $*$-subalgebra $\mathscr{A} \subset A$ such that
$$\Delta(\mathscr{A}) \subset \mathscr{A} \otimes_{\mathrm{alg}} \mathscr{A}$$
and $\big(\mathscr{A}, \Delta\big|_{\mathscr{A}}\big)$ is a **Hopf $*$-algebra** (counit e, antipode κ).
- For $A = \mathrm{C}^*(\Gamma)$ we have $\mathscr{A} = \mathbb{C}[\Gamma]$.
- If $A = \mathrm{C}(G)$ then $\mathscr{A}$ is the span of matrix elements of irreps.

Universal quantum group

- (A, Δ) — compact quantum group, $\mathscr{A}$ — it's Hopf algebra.
- The enveloping C^*-algebra A_u of $\mathscr{A}$ carries a unique comultiplication $\Delta_u : A_u \to A_u \otimes A_u$ such that

$$\begin{array}{ccc} A_u & \xrightarrow{\Delta_u} & A_u \otimes A_u \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho\otimes\rho} \\ A & \xrightarrow[\Delta]{} & A \otimes A \end{array}$$

  where $\rho : A_u \to A$ is the quotient map.
- (A_u, Δ_u) is a compact quantum group — **universal** (A, Δ).
- The Hopf algebra associated with (A_u, Δ_u) is $\mathscr{A}$.
- Also the Hopf algebra associated with (A_r, Δ_r) is $\mathscr{A}$.

Universal quantum group

$$
\begin{array}{ccc}
\mathscr{A} & \subset & A_u \\
\Vert & & \downarrow \lambda \\
\mathscr{A} & \subset & A \\
\Vert & & \downarrow \rho \\
\mathscr{A} & \subset & A_r
\end{array}
$$

Woronowicz characters

- (A, Δ) — compact quantum group, $\mathscr{A}$ — it's Hopf algebra.
- $\exists!$ family $(f_z)_{z\in\mathbb{C}}$ of non-zero multiplicative functionals on $\mathscr{A}$ such that
  - for an $a \in \mathscr{A}$ the function $z \mapsto f_z(a)$ is entire,
  - $f_0 = e, \quad f_{z_1} * f_{z_2} = f_{z_1+z_2},$ $\Big(\psi * \varphi = (\psi \otimes \varphi)\circ\Delta\Big)$
  - $f_{\overline{z}}(a^*) = \overline{f_{-z}(a)}$ for all $a \in \mathscr{A}$, $z \in \mathbb{C}$,
  - $f_z(\kappa(a)) = f_{-z}(a)$ for all $a \in \mathscr{A}$, $z \in \mathbb{C}$,
  - $\kappa^2(a) = f_{-1} * a * f_1$ for all $a \in \mathscr{A}$. $\Big(\psi * a = (\mathrm{id} \otimes \psi)\Delta(a)\Big)$
- $(f_{\mathrm{i}t})_{t\in\mathbb{R}}$ are $*$-characters of $\mathscr{A}$
  $\Rightarrow$ they extend to characters of A_u.
- The family $(f_z)_{z\in\mathbb{C}}$ is related to the modular function on the *dual* of (A, Δ).
- We have $f_z = e$ for all z **iff** the Haar measure is a trace.

Quantum two-torus

- $\theta \in]0,1[, \quad A_\theta = \mathrm{C}^*(u,v)$

$$u^*u = \mathbf{1} = uu^*, \quad v^*v = \mathbf{1} = vv^*, \quad uv = \mathrm{e}^{2\pi \mathrm{i}\theta} vu.$$

- A_θ admits a faithful trace.
- If there is $\Delta : A_\theta \to A_\theta \otimes A_\theta$ such that (A_θ, Δ) is a c.q.g. then
  - the Haar measure of (A_θ, Δ) is a trace,
  - $\kappa^2 = \mathrm{id}$ (i.e. (A_θ, Δ) is a **Kac algebra**). (P.M.S.)
- A_θ is nuclear. Therefore
  - ${A_\theta}_r = {A_\theta}_u$, (This property is called *co-amenability*.)
  - the counit of $\mathscr{A}$ is continuous on A_θ. (Bedos, Murphy & Tuset)
- This means that A_θ must admit a character, but it does not.
- The quantum two-torus is not a quantum group (for $\theta \neq 0$).
- Neither is any higher dimensional *quantum* torus. □

Bratteli-Elliott-Evans-Kishimoto quantum two-spheres

- $C_\theta = C(S^2_\theta)$ is defined as $C_\theta = {A_\theta}^\alpha$, where $\alpha \in \mathrm{Aut}(A_\theta)$

$$\alpha(u) = u^*, \quad \alpha(v) = v^*.$$

- C_θ admits a faithful trace,
- C_θ is nuclear,
- C_θ does not admit a character. □

Standard Podleś quantum two-spheres

- $q \in [-1, 1] \setminus \{0\}, \quad \mathrm{C}(S^2_{q,0}) = \mathscr{K}^+$.
- Assume that there is $\Delta : \mathscr{K}^+ \to \mathscr{K}^+ \otimes \mathscr{K}^+$ such that $(\mathscr{K}^+, \Delta)$ is a c.q.g.
- One can show that it's Haar measure must be faithful.
- $\mathscr{K}^+$ admits a character, and so $(\mathscr{K}^+, \Delta)$ is co-amenable. Thus
  - all Woronowicz characters are continuous,
  - but there is only one character on $\mathscr{K}^+$,
  - so $f_{it} = e$ for all $t \in \mathbb{R}$,
  - so $f_z = e$ for all $z \in \mathbb{C}$,
  - so Haar measure of $(\mathscr{K}^+, \Delta)$ is a trace.
- There are no faithful traces on $\mathscr{K}^+$. □

Natsume-Olsen quantum two-spheres

- $t \in \left[0, \frac{1}{2}\right[, \quad B_t = \mathrm{C}(S_t^2), \quad B_t = \mathrm{C}^*(\zeta, z)$

$$\zeta^*\zeta + z^2 = \mathbf{1} = \zeta\zeta^* + (t\zeta\zeta^* + z)^2,$$
$$\zeta z - z\zeta = t\zeta(\mathbf{1} - z^2).$$

- For $t = 0$ we get $\mathrm{C}(S^2)$ and S^2 is not a group.
- We can show that if there is $\Delta : B_t \to B_t \otimes B_t$ such that (B_t, Δ) is a c.q.g. then
  - The Haar measure of G cannot be a trace,
  - $B_{t\,r}$ possesses a character.
- Thus (B_t, Δ) must be co-amenable, so $B_{t\,r} = B_{t\,u} = B_t$
  $\Rightarrow$ all Woronowicz characters are continuous on B_t.
- But B_t has only two characters (not enough). □