When a quantum space is not a group?

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Plan

Introduction

Compact quantum semigroups and groups

Additional structure

Examples

- Any topological space X can be given a structure of an associative topological semigroup (e.g. $x \cdot y = y$).
- Can [0,1] be given a structure of a topological group?
- The same goes for any manifold with boundary.
- A related question: let

$$X = \left\{ egin{bmatrix} s \ t \ r \end{bmatrix} \in \mathbb{C}^3 \middle| egin{array}{c} st = -r^2, \ |s| + |t| = 1 \end{array}
ight\}$$

and define multiplication on X by

$$\begin{bmatrix} s \\ t \\ r \end{bmatrix} \cdot \begin{bmatrix} s' \\ t' \\ r' \end{bmatrix} = \begin{bmatrix} 2(r\overline{t} - s\overline{r})r' + ss' + \overline{t}t' \\ 2(t\overline{r} - r\overline{s})r' + ts' + \overline{s}t' \\ (|s|^2 - |t|^2)r' + rs' + \overline{r}t' \end{bmatrix}.$$

Is *X* a topological group?

- We will investigate problems of existence of group structure on compact quantum spaces.
- A compact quantum space is an object of the category dual to the category of unital C^* -algebras.
- A compact quantum semigroup is a pair
 - (A, Δ) unital *-homomorphism $A \to A \otimes A$
 - $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$.
- Example: A = C(S) (S compact semigroup),

$$\Delta(f) \in A \otimes A = C(S \times S), \qquad \Delta(f)(s,t) = f(st).$$

 A compact quantum group is a compact quantum semigroup (A, Δ) such that

$$\operatorname{span} \left\{ (a \otimes \mathbf{1}) \Delta(b) \, \middle| \, a, b \in A \right\} \subset_{\operatorname{dense}} A \otimes A,$$
 $\operatorname{span} \left\{ \Delta(a) (\mathbf{1} \otimes b) \, \middle| \, a, b \in A \right\} \subset_{\operatorname{dense}} A \otimes A.$

• In case A = C(S) density conditions correspond to

$$(s \cdot t = s \cdot t') \Rightarrow (t = t'),$$

 $(s \cdot t = s' \cdot t) \Rightarrow (s = s').$

- Example:
 - $A = C^*(\Gamma)$ (Γ discrete group), $\Delta(\gamma) = \gamma \otimes \gamma$ ($\gamma \in \Gamma$).

Haar measure

Theorem (S.L. Woronowicz)

Let (A, Δ) be a compact quantum group. Then there exists a unique state h on A such that

$$(\mathrm{id}\otimes h)\Delta(a)=(h\otimes\mathrm{id})\Delta(a)=h(a)\mathbf{1}$$

for all $a \in A$.

• For A = C(G) (G — compact group)

$$h(f) = \int_G f(t) dt.$$

• For $A = C^*(\Gamma)$ (Γ — discrete group)

$$h(\gamma) = \delta_{\gamma,e}$$

for $\gamma \in \Gamma$. (This might not be faithful.)

Reduced quantum group

- (A, Δ) compact quantum group, h it's Haar measure.
- Let $J = \{a \in A \mid h(a^*a) = 0\}, \quad A_r = A/J, \quad \lambda : A \rightarrow A_r$.
- There is a unique $\Delta_r: A_r \to A_r \otimes A_r$ such that

$$\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \otimes A \\
\lambda \downarrow & & \downarrow \lambda \otimes \lambda \\
A_r & \xrightarrow{\Delta_r} & A_r \otimes A_r
\end{array}$$

- (A_r, Δ_r) is a compact quantum group reduced (A, Δ) .
- For $A = C^*(\Gamma)$ we have $A_r = C_r^*(\Gamma)$.

Hopf algebra

- (A, Δ) compact quantum group.
- There exists a unique dense unital *-subalgebra A ⊂ A such that

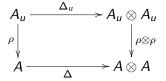
$$\Delta(\mathscr{A}) \subset \mathscr{A} \otimes_{\mathrm{alg}} \mathscr{A}$$

and $(\mathscr{A}, \Delta|_{\mathscr{A}})$ is a **Hopf** *-algebra (counit *e*, antipode κ).

- For $A = C^*(\Gamma)$ we have $\mathscr{A} = \mathbb{C}[\Gamma]$.
- If A = C(G) then $\mathscr A$ is the span of matrix elements of irreps.

Universal quantum group

- (A, Δ) compact quantum group, \mathscr{A} it's Hopf algebra.
- The enveloping C^* -algebra A_{μ} of \mathscr{A} carries a unique comultiplication $\Delta_{\mu}: A_{\mu} \to A_{\mu} \otimes A_{\mu}$ such that

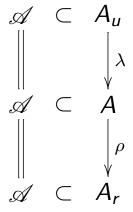


where $\rho: A_u \to A$ is the quotient map.

- (A_u, Δ_u) is a compact quantum group universal (A, Δ) .
- The Hopf algebra associated with (A_{μ}, Δ_{μ}) is \mathscr{A} .
- Also the Hopf algebra associated with (A_r, Δ_r) is \mathscr{A} .

Universal quantum group

Compact quantum semigroups and groups



Woronowicz characters

- (A, Δ) compact quantum group, \mathscr{A} it's Hopf algebra.
- ∃! family (f_z)_{z∈ℂ} of non-zero multiplicative functionals on A such that
 - for an $a \in \mathscr{A}$ the function $z \mapsto f_z(a)$ is entire,

•
$$f_0 = e$$
, $f_{z_1} * f_{z_2} = f_{z_1+z_2}$, $\left(\psi * \varphi = (\psi \otimes \varphi) \circ \Delta\right)$

- $f_{\overline{z}}(a^*) = \overline{f_{-z}(a)}$ for all $a \in \mathscr{A}$, $z \in \mathbb{C}$,
- $f_z(\kappa(a)) = f_{-z}(a)$ for all $a \in \mathscr{A}$, $z \in \mathbb{C}$,
- $\kappa^2(a) = f_{-1} * a * f_1$ for all $a \in \mathscr{A}$. $\Big(\psi * a = (\mathrm{id} \otimes \psi)\Delta(a)\Big)$
- $(f_{it})_{t \in \mathbb{R}}$ are *-characters of \mathscr{A} \Rightarrow they extend to characters of A_u .
- The family $(f_z)_{z\in\mathbb{C}}$ is related to the modular function on the dual of (A, Δ) .
- We have $f_z = e$ for all z iff the Haar measure is a trace.

Quantum two-torus

 $oldsymbol{ heta} heta \in]0,1[$, $A_{ heta} = \mathrm{C}^*(u,v)$

$$u^*u = 1 = uu^*, \quad v^*v = 1 = vv^*, \quad uv = e^{2\pi i\theta}vu.$$

- A_θ admits a faithful trace.
- If there is $\Delta: A_{\theta} \to A_{\theta} \otimes A_{\theta}$ such that (A_{θ}, Δ) is a c.q.g. then
 - the Haar measure of (A_{θ}, Δ) is a trace,

•
$$\kappa^2 = \mathrm{id}$$
 (i.e. (A_θ, Δ) is a Kac algebra). (P.M.S.)

• A_{θ} is nuclear. Therefore

• $A_{\theta r} = A_{\theta m}$

- (This property is called co-amenability.)
- the counit of $\mathscr A$ is continuous on A_{θ} . (Bedos, Murphy & Tuset)
- This means that A_{θ} must admit a character, but it does not.
- The quantum two-torus is not a quantum group (for $\theta \neq 0$).
- Neither is any higher dimensional quantum torus.

Bratteli-Elliott-Evans-Kishimoto quantum two-spheres

• $C_{\theta} = C(S_{\theta}^2)$ is defined as $C_{\theta} = A_{\theta}{}^{\alpha}$, where $\alpha \in Aut(A_{\theta})$

$$\alpha(u)=u^*,\quad \alpha(v)=v^*.$$

- C_{θ} admits a faithful trace,
- C_{θ} is nuclear,
- C_θ does not admit a character.

Examples

Standard Podleś quantum two-spheres

- $q \in [-1,1] \setminus \{0\}, \quad \mathsf{C}(S_{a,0}^2) = \mathscr{K}^+.$
- Assume that there is $\Delta: \mathcal{K}^+ \to \mathcal{K}^+ \otimes \mathcal{K}^+$ such that (\mathcal{K}^+, Δ) is a c.q.g.
- One can show that it's Haar measure must be faithful.
- \mathcal{K}^+ admits a character, and so (\mathcal{K}^+, Δ) is co-amenable. Thus
 - all Woronowicz characters are continuous,
 - but there is only one character on \(\mathcal{K}^+ \),
 - so $f_{it} = e$ for all $t \in \mathbb{R}$,
 - so $f_z = e$ for all $z \in \mathbb{C}$,
 - so Haar measure of (\mathcal{K}^+, Δ) is a trace.
- There are no faithful traces on K⁺.

Natsume-Olsen quantum two-spheres

•
$$t \in [0, \frac{1}{2}[, B_t = C(S_t^2), B_t = C^*(\zeta, z)]$$

$$\zeta^* \zeta + z^2 = \mathbf{1} = \zeta \zeta^* + (t\zeta \zeta^* + z)^2,$$

$$\zeta z - z \zeta = t \zeta (\mathbf{1} - z^2).$$

- For t = 0 we get $C(S^2)$ and S^2 is not a group.
- We can show that if there is $\Delta: B_t \to B_t \otimes B_t$ such that (B_t, Δ) is a c.q.g. then
 - The Haar measure of G cannot be a trace,
 - B_{tr} possesses a character.
- Thus (B_t, Δ) must be co-amenable, so $B_{tr} = B_{tu} = B_t$ \Rightarrow all Woronowicz characters are continuous on B_t .
- But B_t has only two characters (not enough).