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Isolated Horizons and Their Secrets.

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- Geometry and invariants of non-expanding horizons
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  -invariant frame,
  -crossover spheres of WIHs and the crossover sphere

- Isolated Horizons
  -When a non-expanding horizon is isolated?
  -The uniqueness issue -special IHs of 2-dimensional symmetry group
  -the Kerr IH: invariant characterizations

- Neighborhood of IH
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- Open Problems
  -Sufficient conditions for the existence of Killing vector fields
  -the degenerate cases,
  -absorbing horizons.

Geometry and invariants of non-expandig horizons

Definitions and assumptions.

Definition: *Non-expanding horizon* $\triangle$ *is a null 3-submanifold of* M *such that:*

$i)$ $\triangle$ *is diffeomorphic to the product*

$$\hat{\triangle} \times \mathbf{R}, \tag{1}$$

where $\hat{\triangle}$ *is a 2-sphere, and the fibers of the projection*

$$\hat{\triangle} \times \mathbf{R} \to \hat{\triangle} \tag{2}$$

correspond to the null geodesics in $\triangle$*;*

$ii)$ *the family of null geodesics tangent to* $\triangle$ *is non-expanding.*

The energy assumptions and consequences. Assumptions:

$$T_{\mu\nu}\ell^\mu\ell^\nu \geq 0, \quad T_{\mu\nu}\ell^\nu \quad \text{causal} \tag{3}$$
$$\text{for} \quad \ell, X \in T(\triangle), \quad \text{s.t. } \ell^\mu\ell_\mu = 0. \tag{4}$$
$$\tag{5}$$

Consequences:

$$R_{a\nu}\ell^\nu X^a = 0, \quad X \in T(\triangle). \tag{6}$$

And ℓ is a double principal direction of the Weyl tensor.

Geometry of $\triangle$. The induced metric q satisfies

$$\mathcal{L}_\ell q \ = \ 0 \tag{7}$$

It implies, that ∇ preserves the tangent bundle $T(\triangle)$, and defines a covariant derivative therein,

$$\mathcal{D}_X Y \ := \ \nabla_X Y. \tag{8}$$

Definition: The geometry of $\triangle$ is the pair $(q, \mathcal{D})$.

$\mathcal{D}$ is not determined by q, $[\mathcal{L}_\ell, \mathcal{D}] \neq 0$ in general.

Rotation of $\triangle$, the geometry ingreedient. There is a 1-form ω on $\triangle$ s. t.

$$\mathcal{D}\ell \;=\; \omega \otimes \ell \tag{9}$$

Definition: ω *is called the rotation potential 1-form of* $(\triangle, \ell)$, *and the quantity*

$$\kappa^{(\ell)} \;=\; \ell^a \omega_a \tag{10}$$

the surface gravity of ℓ.

$$\ell' = f\ell \quad \Rightarrow \quad \omega' = \omega + d\ln f \tag{11}$$

The invariant 2-form is,

$$d\omega = 2i\mathrm{Im}\Psi_2 \mathrm{vol}^{(2)}, \quad D\mathrm{Im}\Psi_2 \;=\; 0. \tag{12}$$

can be considered as the angular velocity of $\triangle$. *(blackboard!) The zeroth law:*

$$d\kappa^{(\ell)} \hat{=}\; \mathcal{L}_\ell \omega \tag{13}$$

This is a geometric version of the zeroth law of BH termodynamics.

The remaining ingreedients of $\mathcal{D}$. *Deformations of the 2-sphere sections* S_v *of* $\triangle$ *as expanded in the* *orthogonal-transversal null direction define the remaining components of $\mathcal{D}$, so called shear and expansion. Those depend on the pullback of $Ricci$ tensor on S_v, and on initial conditions at any fixed crossection of $\triangle$.*

<u>Invariant choice of ℓ</u>. For every geometry $(q, \mathcal{D})$, there is ℓ, s.t.

$$d\kappa^{(\ell)} \ = \ \mathcal{L}_\ell \omega \ = 0, \tag{14}$$

but it is not unique. We call $(\triangle, \ell)$ a Weakly IH. BLACKBOARD Proposition:For a given value of the surface gravity

$$\kappa^{(\ell)} \ = \ \kappa_0 \neq 0, \tag{15}$$

every generic non-expanding horizon $(\triangle, q, \mathcal{D})$ such that

$$DR_{m\bar{m}} = 0 \tag{16}$$

admites a unique null, non-trivial WIH structure $(\triangle, \ell)$ s. t. for every section S_v of $\triangle$ it is true *that the transport of S_v by the flow of ℓ does not change the corresponding transversal expansion μ. The genericity condition is that* 0 *is not an eigen value of the following operator $M : L^2(S) \to L^0(S)$,*

$$M \ := \ \hat{\Delta} + 2\hat{\omega}^A \partial_A + \hat{\mathrm{div}}\hat{\omega} + \hat{\omega}_A \hat{\omega}_B \hat{q}^{AB} - K + R_{m\bar{m}}, \quad (17)$$

and the non-triviality is that ℓ can not be identically 0 *on a null geodesic.*

Geometric version of the proposition: crossover sphere

Consider the analytic-geodesic completion of $\triangle$ and $(q, \mathcal{D})$. For every ℓ such that

$$\kappa^{(\ell)} \ = \ \kappa_0 \neq 0, \quad (18)$$

The equation

$$\ell \ = \ 0, \quad (19)$$

defines a crossection S_ℓ of the completion.

Proposition: A null vector field tangent to $\triangle$ is the invariant vector field ℓ of the previous proposition if *and only if* $\kappa^{(\ell)} = \kappa_0$ *and the zero set of* ℓ *has zero expansion.*

We will call such crossection the crossover sphere of $\triangle$.

This solves the issue of invariant evolution of the data: it is

$$[\mathcal{L}_\ell, \mathcal{D}]^a{}_{bc} \;=\; \ell^a(\dot{\lambda} m_b m_c + c.c). \tag{20}$$

Good cuts foliation. *Given* $(q, \mathcal{D})$ *consider a space-like foliation of* $\triangle$ *preserved by the flow of the invariant vector field* ℓ. *The pullback of the rotation 1-form potential* ω *to the slices, defines a unique 1-form* $\hat{\omega}$ *on* S,

$$\hat{\omega} = \hat{*}\hat{d}U + \hat{d}p. \tag{21}$$

The first term is foliation invariant, the second is arbitrary.

DefinitionWe call a foliation the good cuts foliation if

$$\widehat{\omega} = \widehat{*}\widehat{d}U. \tag{22}$$

For every ℓ *such that* $\kappa^{(\ell)} = \mathrm{const}$, *there is a unique good cuts foliation.*

Invariant frame on $\triangle$*.* *The invariant* ℓ *and corresponding good cuts foliation uniquely define a transversal, null vector field* n, *such that*

$$n_\mu \ell^\mu = -1, \quad n \ \text{orthogonal to the foliation.} \tag{23}$$

Finally, we can complete (n, ℓ) *to a null frame* $(m, \bar{m}, n, \ell)$ *by using the Gauss curvature* K *of the crossections, wherever* $dK \neq 0$,

$$(\delta - \overline{\delta})K \ = \ 0. \tag{24}$$

Proposition The above invariant null frame is uniquely and naturally defined by the geometry $(q, \mathcal{D})$*. (diffeos: blackboard!)*

Isolated Horizons.

Definition A non-expanding horizon $(\triangle, q, \mathcal{D})$ and a null, tangent vector field ℓ are called IH if

$$[\mathcal{L}_\ell, \mathcal{D}] \;=\; 0. \tag{25}$$

The existence Issue:

(i) A non-expanding horizon $(\triangle, q, \mathcal{D})$ admites the IH structure if and only if the invariant vector field ℓ defines it.

(ii) We have also derived the explicite existence conditions.

True degrees of fredom In the $\kappa^{(\ell)} \neq 0$ case, the geometry $(q, \mathcal{D})$ is explicitely given by: Gauss curvature K, the rotation potential U, the Ricci tensor components given by contraction with the vectors tangent to $\triangle$.

The crossover sphere *Proposition:If* $(\triangle, [\ell])$ *is an isolated horizon, then the crossover sphere of the completion of* $\triangle$ *has zero shear and expansion in any orthogonal null direction. Conversely, if the completion of a non-expanding and shear free horizon* $\triangle$ *containes a crossection that is shear free and non-expanding in every orthogonal null direction, then, the corresponding null vector field* ℓ *is an isolated horizon.*

The uniqueness issue *Given an isolated horizon* $(\triangle, q, \mathcal{D})$*, are there* ℓ *and* $\ell' = f\ell$ $(df \neq 0)$ *such that*

$$[\mathcal{L}_\ell, \mathcal{D}] = 0 = [\mathcal{L}_{\ell'}, \mathcal{D}].? \tag{26}$$

Results: (i) The general case: we derived necessary non-uniqueness conditions (a non-uniqueness test).

(ii) Special cases in which the uniqueness has been proven:

(a) If the geometry of an isolated horizon is sufficiently close to that of the Kerr-Nrewman horizon

(b) If an isolated horizon $\triangle$ is non-rotating and the Ricci tensor vanishes on $\triangle$

(c) If the Ricci tensor vanishes on an isolated horizon $\triangle$, and the geometry $(q, \mathcal{D})$ is not the one characterized below.

The isolated horizon of 2-dim null symmetry group.
The only non-unique IH structures are define on the following non-expanding horizon $(\triangle, q, \mathcal{D})$. Consider a vacuum isolated horizon $(\triangle, q, \mathcal{D})$ and its null symmetry $\ell = \ell_0$ such that

$$\kappa^{(\ell)} = 0 \tag{27}$$

and it admites a crossection of the zero expansion and shear in each orthogonal null direction. This horizons geometry $(q, \mathcal{D})$ has 2- dimensional group of null symmetries.

Proposition (Pawlowski):In the cylindrically symmetric case, the only non-unique IH structures come from the modification of the extemal Kerr data.

<u>Geometric characterization of the Kerr horizon.</u>

Proposition:Each of the below conditions 1) and 2) is necessary and sufficient for an axially symmetric IH to be the Kerr IH:

1. Let Φ *be the rotation Killing vector,*

$$d({\Psi_2}^{-\frac{1}{3}}) = a_0 \Phi \lrcorner \epsilon^{(2)} \tag{28}$$

2. $R_{\mu\nu}|_{\triangle} = \partial R_{\mu\nu}|_{\triangle}$ *and the Weyl tensor is of the type* D *at* $\triangle$*.*

Neighborhood of IH

The unique extension of the good cuts foliation, invar

Given a horizon (non-expanding or isolated), the invariant null vector field ℓ, and the corresponding good cuts foliation we extend it in the following way: (blackboard)

(i) the good cuts foliation by the family of the orthogonal null geodesics

(ii) The vector field n, by

$$\nabla_n n \;=\; 0, \tag{29}$$

(iii) The vector field ℓ, to ξ, s.t

$$\xi|_{\triangle} = \ell,\ \mathcal{L}_n \xi = 0. \tag{30}$$

The vector field ξ is not any longer null out of $\triangle$.

PropositionIf $\triangle$ is a Killing horizon, then ξ is the Killing vector field.

The unique extension of the null frame By

$$\nabla e^{\mu} \;=\; 0. \tag{31}$$

The Gauss coordinates A coordinates system (x^A, v) compatible with the good cuts foliation is naturally extended by

$$n^{\mu}\partial_{\mu}x^A = n^{\mu}\partial_{\mu}v \;=\; 0 \tag{32}$$

and completed by r,

$$n^{\mu}\partial_{\mu}r \;=\; -1, \quad r|_{\triangle} \;=\; 0. \tag{33}$$

to coordinates (x^A, v, r).

The invariants:

(i) The curvature of the 2-spheres $r = \text{const}$, $v = \text{const}$

(ii) the products $\xi^{\mu}\ell_{\mu}$, $\xi^{\mu}n_{\mu}$, $\xi^{\mu}m_{\mu}$

(iii) Any geometric tensor components.

The degrees of freedom: Friedrich's reduced data.
The degrees of freedom (q, U) *in the geometry* $(q, \mathcal{D})$ *are compatible with the Friedrich-Rendall reduced data and provide true degrees of freedom for a neighborhood of* $\triangle$*. In the* $\kappa^{(\ell)} \neq 0$*, vacuum case, they are:*

(i) Geometry $(q, \mathcal{D})$ *of* $\triangle$ *freely parametrized by* q, U

(ii) R_{nmnm} *on two transversal half-leaves of the invariant null foliation.*

Symmetric horizons and the existence of the Killing vector fields conditions.

Symmetries of $\triangle$*.*

Definition A vector field $k \in T(\triangle)$ *generates a symmetry of* $\triangle$*, if*

$$\mathcal{L}_k q \;=\; 0, \quad L_k \mathcal{D} \;=\; 0.$$

Proposition The invariant null vector field ℓ, the corresponding good cuts foliation, and the ingredients we use to describe $(q, \mathcal{D})$ are preserved by the symmetries of $\triangle$.

Every symmetry generating $k \in T(\triangle)$ is given by a Killing vector field of a sigle slice, and a constant a_0,

$$k \ = \ a_0\ell + Y, \quad \mathcal{L}_\ell Y = 0, \mathcal{L}_Y q \ = \ 0.$$

It follows from the evolution of $\mathcal{D}$ along the null generators, that:

PropositionSuppose $k = a_0\ell + Y$; then then both the vector fields $a_0\ell$ and Y generate symmetries of $\triangle$. (blackboard)

Classification of the symmetries of $\triangle$:

Class 1. $k = \ell$. Isolated horizon

Class 2. $k = Y$. *Axially symmetric non-expanding horizon.*

Class 3. $k_1 = \ell$, $k_2 = Y$, *Axially symmetric isolated horizon*

Class 4. $k_1 = \ell$ *and 3-dim rotation symmetry group.*

<u>*Killing vector fields: possible candidates.*</u>

PropositionSuppose K *is a Killing vector field tangent to* $\triangle$ *and defined in a neighborhood of* $\triangle$. *Then:*

1.K *preserves the invariant foliation of the neighborhood, the vector field* ξ *and the invariant null frame.*

2. K *is uniquely determined by a symmetry* k *of* $\triangle$ *it generates, by*

$$\mathcal{L}_n K \ = \ 0, \quad K_{|\triangle} \ = \ k.$$

Therefore, given a non-expanding horizon $\triangle$, the natural null vector field ℓ, the good cuts foliation and the corresponding Gauss coordinates, the only possible candidates for a Killing vector field are,

$$K = \begin{cases} a_0 \partial_v = \xi, \\ b_0 \partial_\varphi, \\ a_0 \partial_v + b_0 \partial_\varphi, \end{cases} \tag{34}$$

the 2-sphere $\hat{\triangle}$ of the null generators cordinates $(x^A) = (\theta, \varphi)$ are adjusted to a Killing vector field of $\hat{\triangle}$.

Killing vector fields: the existence conditions, the vacu

1. Axial symmetry:

(i) Necessary condition at $\triangle$: the axial symmetry of $(q, \mathcal{D})$

(ii) Necessary and sufficient: the axial symmetry of the reduced data. trivial

2. The Killing horizon case:

(i) Necessary conditions at $\triangle$: the functions $\partial_r^N R_{nmnm}$, $N = 0, 1, 2, 3, 4, ...$ at a fixed crossection of $\triangle$ are determined by $(q, \mathcal{D})$ in an explicite way.

(ii) Necessary and sufficient: the following equation to be satisfied by $\Psi_4 := \overline{R_{nmnm}}$ on a fixed lief of the invariant foliation,

$$(X^A \partial_A + H\partial_r)\Psi_4 = (\rho - 4\epsilon)\Psi_4 + (\bar{\delta} + 4\pi + 2\alpha)\Psi_3 - 3\lambda\Psi_2.$$

3. The cross diagonal case ($K = a_0 \ell + \partial_\varphi$)

The necessary conditions: both, the necessary conditions 1. and 2. above.

Since R_{mnmn} is free, in this way we control the non-expanding (isolated) horizons which do NOT admit ani Killing vector field.