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Logarithmic two-loop corrections to the Lamb shift in hydrogen

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Abstract

Higher order $(\alpha/\pi)^2\,(Z\,\alpha)^6$ logarithmic corrections to the hydrogen Lamb shift are calculated. The results obtained show the two-loop contribution has a very peculiar behavior, and significantly alter the theoretical predictions for low lying S-states.

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The calculation of the two-loop contribution to the Lamb shift in hydrogen is one of the most challenging projects in bound state QED [1,2]. Since direct numerical calculations with the use of Dirac-Coulomb propagators have not yet been completed, one has to rely on the $Z\,\alpha$ expansion:

$$\Delta E = m\left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^4 \Big\{B_{40} + (Z\,\alpha)\,B_{50} \\ +(Z\,\alpha)^2\left[\ln^3(Z\,\alpha)^{-2}\,B_{63} + \ln^2(Z\,\alpha)^{-2}\,B_{62} + \ln(Z\,\alpha)^{-2}\,B_{61} + B_{60}\right] + \ldots\Big\}. \tag{1}$$

The leading order correction B_{40} can be obtained from the slope of the electron form-factors F_1 and F_2 at $q^2 = 0$. It is known analytically and its numerical value is quite small (for S-states including vacuum polarization)

$$B_{40} = 0.538941\,. \tag{2}$$

The calculation of the next order correction B_{50} was completed only a few years ago independently by two groups in [3] and [4]. The value was surprisingly large

$$B_{50} = -21.5561(31)\,. \tag{3}$$

Moreover, this correction led to strong disagreement in He^+ Lamb shift with the most precise experimental value in [5], while for hydrogen Lamb shift, led to agreement with the Mainz value for the proton charge radius [6]. This large value of B_{50} compared to B_{40} indicates a very slow convergence or even might suggest a nonperturbative behavior of the two-loop contribution. Indeed, the direct numerical calculations of one diagram, the loop-by-loop electron self–energy by Mallampalli and Sapirstein in [7], shows that the value of this correction at $Z = 1$ is of different sign and magnitude, that the one based on the first two terms of analytic expansion. Moreover, this numerical calculation was in disagreement with the analytical value of B_{63} in [8], while it was argued in [8], that this correction comes only from this one diagram in the covariant gauge. A year later another group [9] calculated numerically this one diagram and found agreement with the analytic expansion including $\ln^3(Z\,\alpha)^{-2}$ term. While this may suggest that the first numerical calculations were incorrect, a very recent, third numerical result by Yerokhin in [10] confirmed the first one [7]. So, this situation with the two-loop contribution is very unclear. Moreover, the analytic value $\ln^3(Z\,\alpha)^{-2}$ term corresponding to all diagrams, was confirmed independently by several groups, so this situation is even more confusing. It was argued, by Yerokhin in [10], that the $\ln^3(Z\,\alpha)^{-2}$ term for this one loop-by-loop diagram is different from the total value of B_{63}, and in fact found an additional contribution. However, the value for this term, coming from all diagrams might be correct, because other diagrams may contain compensating terms. The goal of this work is to shed some light into higher order two-loop corrections and calculate all logarithmic terms: B_{63}, B_{62}, and B_{61}. We find that indeed the two-loop contribution has a very peculiar behavior, as the higher order term B_{61} dominates and reverses the sign for the overall logarithmic contribution. In next sections we present some details of this calculation. First, a simple example is worked out to demonstrate the method, then we pass to the most difficult two-photon-loop diagrams and complete with remaining diagrams containing an electron loop. Conclusions with prospects of calculation of B_{60} summarize this work.

I. SIMPLE EXAMPLE

The example to demonstrate the calculational method is the asymptotic expansion of

$$P(\omega) \equiv \langle\phi|\,\boldsymbol{p}\,\frac{1}{E-(H+\omega)}\,\boldsymbol{p}\,|\phi\rangle \tag{4}$$

$$= -\frac{1}{\omega}+\frac{2}{\omega^2}-\frac{4\sqrt{2}}{\omega^{\frac{5}{2}}}+\frac{4-12\,\ln(2)+4\,\ln(\omega)}{\omega^3}+\dots \tag{5}$$

around large ω for ground state of the hydrogen atom. More precisely, we concentrate on the ω^{-3} term. For simplicity, we put here $m=1, \alpha=1$. From one side $P(\omega)$ is known analytically [11]

$$P(\omega) = -\frac{384\,\tau^5}{(1+\tau)^8\,(2-\tau)}\,{}_2F_1(4,2-\tau,3-\tau,\zeta)\,, \tag{6}$$

where

$$\zeta = \left(\frac{1-\tau}{1+\tau}\right)^2,\;\; \tau = \frac{1}{\sqrt{2\,(\omega+1/2)}}\,, \tag{7}$$

so one could get this coefficient from here. However, our final goal is to calculate the two-loop contribution, for which no analytic formula has been derived so far. Therefore, we use a different approach, based on the effective Hamiltonian. First, we regularize the Coulomb interaction by the following replacement:

$$V(r) = -\frac{1}{r} \to -\frac{1}{r}\,(1-e^{-\lambda\,r})\,. \tag{8}$$

With the regularized potential ($P\to P_R$) one can expand P_R in $(H-E)/\omega$ which leads to

$$P_R = -\frac{1}{\omega^3}\,\langle\phi|\boldsymbol{p}\,(H-E)^2\,\boldsymbol{p}|\phi\rangle = -\frac{1}{\omega^3}\,\langle\phi|V'(r)^2|\phi\rangle\,, \tag{9}$$

$$\langle\phi|V'(r)^2|\phi\rangle = 2\,\lambda+8\,\ln(3)-8\,\ln(\lambda)-2\,, \tag{10}$$

where the last expectation value is taken from [12]. The remaining part, which was left out by this replacement, is obtained from the subtracted forward scattering amplitude. Two photon exchange is

$$P_2 = \int\frac{d^3p}{(2\,\pi)^3}\,64\,\pi\left[\frac{\boldsymbol{p}}{p^4}\,\frac{(-1)}{p^2/2+\omega}\,\frac{\boldsymbol{p}}{p^4}-\frac{\boldsymbol{p}}{p^4}\,\frac{\lambda^2}{p^2+\lambda^2}\,\frac{(-1)}{p^2/2+\omega}\,\frac{\lambda^2}{p^2+\lambda^2}\,\frac{\boldsymbol{p}}{p^4}\right] = \frac{2\,\lambda}{\omega^3}\,, \tag{11}$$

where we keep only the ω^{-3} term (ω^{-1} and ω^{-2} are subtracted out before the integration). The three photon exchange requires more subtractions. One Coulomb exchange between photon vertices gives P_{3A}

$$\begin{aligned}P_{3A} = {}& \int\frac{d^3p}{(2\,\pi)^3}\int\frac{d^3p'}{(2\,\pi)^3}\,64\,\pi\left[\frac{\boldsymbol{p}'}{p'^4}\,\frac{(-1)}{p'^2/2+\omega}\,\frac{(-4\,\pi)}{q^2}\,\frac{(-1)}{p^2/2+\omega}\,\frac{\boldsymbol{p}}{p^4}\right.\\ &\left.-\frac{\boldsymbol{p}'}{p'^4}\,\frac{\lambda^2}{p'^2+\lambda^2}\,\frac{(-1)}{p'^2/2+\omega}\,\frac{(-4\,\pi)}{q^2}\,\frac{\lambda^2}{\lambda^2+q^2}\,\frac{(-1)}{p^2/2+\omega}\,\frac{\lambda^2}{p^2+\lambda^2}\,\frac{\boldsymbol{p}}{p^4}\right]\\ ={}& \frac{4\,\ln\omega-8\,\ln\lambda-8\,\ln 3+20\,\ln 2}{\omega^3}\,.\end{aligned} \tag{12}$$

Coulomb exchanges out of photon vertices gives P_{3B}

$$
\begin{aligned}
P_{3B} &= -2048\,\pi^2 \int \frac{d^3 p}{(2\,\pi)^3} \int \frac{d^3 p'}{(2\,\pi)^3} \biggl(\frac{1}{p'^4}\,\frac{1}{q^2}\,\frac{1}{p^2+2\,\omega}\,\frac{1}{p^4} \\
&\quad -\frac{1}{p'^4}\,\frac{\lambda^2}{\lambda^2+p'^2}\,\frac{1}{q^2}\,\frac{\lambda^2}{\lambda^2+q^2}\,\frac{1}{p^2+2\,\omega}\,\frac{\lambda^2}{\lambda^2+p^2}\,\frac{1}{p^4}\biggr) \\
&= \frac{2-32\,\ln(2)+16\,\ln(3)}{\omega^3}\,.
\end{aligned} \tag{13}
$$

There is an implicit subtraction at $p'=0$ for removal of small p' divergence. It corresponds to subtraction of lower order contributions. Additionally, only the ω^{-3} term is selected. The sum

$$
P = P_R + P_2 + P_{3A} + P_{3B} = \frac{4-12\,\ln(2)+4\,\ln(\omega)}{\omega^3} \tag{14}
$$

is independent of λ in the limit of large λ and agrees with that from the expansion of analytic formula in Eq. (5). The advantage of this method is the direct application to the two-loop Lamb shift.

II. TWO-LOOP LAMB SHIFT

The calculations of the two-loop Lamb shift in the order of $\alpha^2 (Z\,\alpha)^6$ is more complicated due to the presence of powers of $\ln(Z\,\alpha)$. It reflects the fact that several energy and momentum regions contribute. For these calculations we introduce a number of cutoff parameters to separate different regions and calculate them independently. In Fig. 1 the integration region of two photon energies ω_1 and ω_2 is split with the help of $\epsilon_1, \epsilon_2, \epsilon'_1, \epsilon'_2$. Additionally λ 'splits' the integration over electron momenta. The splitting itself, does not help too much. The key trick is the assumption that after expansion in $Z\,\alpha$ one goes to the limits $\epsilon_2 \to 0, \epsilon_1 \to 0, \epsilon'_2 \to 0, \epsilon'_1 \to 0, \lambda \to \infty$, in the order as written. The two-loop contribution is split accordingly

$$
\Delta E = E_L + E_M + E_F + E_H\,, \tag{15}
$$

and calculated separately, each term in the most convenient gauge. In the following sections we calculate all logs. The constant term B_{60} is left unevaluated, however we lay the groundwork for its calculation.

III. CONTRIBUTION E_L

Diagrams in the Coulomb gauge in NRQED are presented in Fig. 2. We calculate them first, for photon energies inside a rectangular box $0<\omega_1<\epsilon_1, 0<\omega_2<\epsilon_2, \epsilon_2 << \epsilon_1$, and after combine to the region E_L as shown in Fig. 1. The expression derived from nonrelativistic QED for all these diagrams is:

$$
\begin{aligned}
\mathcal{E}_L = {} & \left(\frac{2\,\alpha}{3\,\pi\,m^2}\right)^2 \int_0^{\epsilon_1} d\omega_1\,\omega_1 \int_0^{\epsilon_2} d\omega_2\,\omega_2 \\
& \Big\{ \langle\phi|p^i\,\frac{1}{E-(H+\omega_1)}\,p^j\,\frac{1}{E-(H+\omega_1+\omega_2)}\,p^i\,\frac{1}{E-(H+\omega_2)}\,p^j|\phi\rangle \\
& +\frac{1}{2}\,\langle\phi|p^i\,\frac{1}{E-(H+\omega_1)}\,p^j\,\frac{1}{E-(H+\omega_1+\omega_2)}\,p^j\,\frac{1}{E-(H+\omega_1)}\,p^i|\phi\rangle \\
& +\frac{1}{2}\,\langle\phi|p^i\,\frac{1}{E-(H+\omega_2)}\,p^j\,\frac{1}{E-(H+\omega_1+\omega_2)}\,p^j\,\frac{1}{E-(H+\omega_2)}\,p^i|\phi\rangle \\
& +\langle\phi|p^i\,\frac{1}{E-(H+\omega_1)}\,p^i\,\frac{1}{(E-H)'}\,p^j\,\frac{1}{E-(H+\omega_2)}\,p^j|\phi\rangle \\
& -\frac{1}{2}\,\langle\phi|p^i\,\frac{1}{E-(H+\omega_1)}\,p^i|\phi\rangle\,\langle\phi|p^j\,\frac{1}{[E-(H+\omega_2)]^2}\,p^j|\phi\rangle \\
& -\frac{1}{2}\,\langle\phi|p^i\,\frac{1}{E-(H+\omega_2)}\,p^i|\phi\rangle\,\langle\phi|p^j\,\frac{1}{[E-(H+\omega_1)]^2}\,p^j|\phi\rangle \\
& +m\,\langle\phi|p^i\,\frac{1}{E-(H+\omega_1)}\,\frac{1}{E-(H+\omega_2)}\,p^i|\phi\rangle \\
& -\frac{m}{\omega_1+\omega_2}\langle\phi|p^i\,\frac{1}{E-(H+\omega_2)}\,p^i|\phi\rangle-\frac{m}{\omega_1+\omega_2}\langle\phi|p^i\,\frac{1}{E-(H+\omega_1)}\,p^i|\phi\rangle\Big\}\,. \qquad (16)
\end{aligned}
$$

It is a two-loop analog of Bethe logs. We have not found a way to calculate its matrix elements analytically in a compact form, therefore we proceed in a different way. One finds, that $\mathcal{E}_L$ as in Eq. (16) depends on α only through ϵ_1 and ϵ_2:

$$
\mathcal{E}_L = \mathcal{E}_L\Big(\frac{\epsilon_1}{\alpha^2},\frac{\epsilon_2}{\alpha^2}\Big)\,. \qquad (17)
$$

To find the logarithmic dependence, we differentiate $\mathcal{E}_L$ over ϵ_1 and ϵ_2 which with the help of $\epsilon_2 << \epsilon_1$ leads to a much simpler expression. The first derivative leads to

$$
\epsilon_1\,\frac{\partial\mathcal{E}_L}{\partial\epsilon_1} = \left(\frac{2\,\alpha}{3\,\pi\,m^2}\right)^2 \int_0^{\epsilon_2} d\omega_2\,\omega_2\,\delta_{\pi\,\delta^3(r)}\,\langle\phi|p^i\,\frac{1}{E-(H+\omega_2)}\,p^i|\phi\rangle\,, \qquad (18)
$$

where $\delta_{\pi\,\delta^3(r)}$ denotes first order corrections to ϕ, H, E due to $\pi\,\delta^3(r)$ operator. This integral was considered and calculated in the context of hyperfine splitting in hydrogen-like systems [13], since the Fermi spin-spin interaction is also proportional to $\delta^3(r)$. The result from that paper which is extended here to any value of principal quantum number is:

$$
\frac{2\,\alpha}{3\,\pi\,m^2}\,\delta_{\pi\,\delta^3(r)} \int_0^{\epsilon} d\omega\,\omega\,\langle\phi|p^i\,\frac{1}{E-(H+\omega)}\,p^i|\phi\rangle = \frac{\alpha}{\pi}\,\alpha^2\,\frac{F(n)}{n^3}\,, \qquad (19)
$$

$$
F(n) = -\frac{2}{3}\,\ln^2\bar\epsilon + \ln\bar\epsilon\left[2\,(1-2\,\ln(2)) + \frac{8}{3}\left(\frac{3}{4}+\frac{1}{4\,n^2}-\frac{1}{n}-\ln(n)+\Psi(n)+C\right)\right] + N(n)\,, \qquad (20)
$$

where N has been calculated only for $n=1$

$$
N \equiv N(1) = 17.8299093\,, \qquad (21)
$$

and $\Psi = \Gamma'/\Gamma$ with Euler Γ function and Euler C constant

$$\Psi(1) = -C; \quad \Psi(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} - C \tag{22}$$

We have introduced here a notation $\bar{\epsilon} = \epsilon/\alpha^2$, which is to be used throughout this work. The result for $n = 1$ with $\mathcal{E} = m\,(\alpha/\pi)^2\,\alpha^6$ is:

$$\epsilon_1\,\frac{\partial \mathcal{E}_L}{\partial \epsilon_1} = \mathcal{E}\,\frac{2}{3}\left[-\frac{2}{3}\,\ln^2(\bar{\epsilon}_2) + 2\,(1 - 2\,\ln 2)\,\ln(\bar{\epsilon}_2) + N\right]. \tag{23}$$

The second derivative, over ϵ_2, is little more difficult to calculate:

$$\begin{aligned}
\epsilon_2\,\frac{\partial \mathcal{E}_L}{\partial \epsilon_2} &= \left(\frac{2\,\alpha}{3\,\pi\,m^2}\right)^2 \int_0^{\epsilon_1} d\omega_1\,\omega_1\,\epsilon_2^2\,\Big\{\ldots\Big\} \\
&= \left(\frac{2\,\alpha}{3\,\pi\,m^2}\right)^2 \left(\int_0^{\epsilon_1'} + \int_{\epsilon_1'}^{\epsilon_1}\right) d\omega_1\,\omega_1\,\epsilon_2^2\,\Big\{\ldots\Big\} = A + B\,.
\end{aligned} \tag{24}$$

One splits it into two parts, with the assumption $\epsilon_1' << \epsilon_2$. The first term A has the same form as that in Eq. (23) with ϵ_2 replaced by ϵ_1'. The second term B is in turn split into two parts $B = B_L + B_H$, where B_L is calculated with the regularized Coulomb potential, as in Eq. (8). One can expand here in the ratio $(H - E)/\omega$ which leads to the expression:

$$B_L = \frac{\mathcal{E}}{9}\,\ln\!\left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}_1'}\right)\left\{\langle\phi|4\,\pi\,\delta^3_\lambda(r)\,\frac{1}{(E-H)'}\,4\,\pi\,\delta^3_\lambda(r)|\phi\rangle + \frac{1}{2}\,\langle\phi|\nabla^2\,4\,\pi\,\delta^3_\lambda(r)|\phi\rangle\right\}. \tag{25}$$

Both terms in above braces have already been calculated in the context of positronium energy levels in [12]

$$\begin{aligned}
\langle\phi|4\,\pi\,\delta^3_\lambda(r)\,\frac{1}{(E-H)'}\,4\,\pi\,\delta^3_\lambda(r)|\phi\rangle &= -\frac{8}{n^3}\left[\frac{\lambda}{2} + 2\,\ln\frac{\lambda}{3} + 8\,\ln\frac{3}{4} - \frac{3}{2} + \frac{2}{n}\right. \\
&\qquad \left. + 2(\ln(n) - \Psi(n) - C)\right],
\end{aligned} \tag{26}$$

$$\langle\phi|\nabla^2\,4\,\pi\,\delta^3_\lambda(r)|\phi\rangle = -\frac{8}{n^3}\left[-\frac{1}{n^2} + \lambda - 4 + 6\,\ln\frac{3}{4}\right], \tag{27}$$

with $n = 1$ in our case. B_H is the difference between B and B_L. In this difference only large electron momenta contribute, therefore it could be obtained in the scattering amplitude approximation, in the same way as P_2 and P_3 in a simple example in the previous section. The result is

$$\begin{aligned}
B_H = \mathcal{E}\,\frac{4}{9}\Bigg[&8 + 5\,\pi^2 - \ln\!\left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}_1'}\right) + 2\,\lambda\,\ln\!\left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}_1'}\right) - 50\,\ln(2)\,\ln\!\left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}_1'}\right) + 18\,\ln(3)\,\ln\!\left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}_1'}\right) \\
&+ \ln\!\left(\frac{\bar{\epsilon}_1'}{\bar{\epsilon}_2}\right)^2 + 4\,\ln\!\left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}_1'}\right)\,\ln\!\left(\frac{\lambda}{\sqrt{\bar{\epsilon}_2}}\right)\Bigg].
\end{aligned} \tag{28}$$

The complete B term is

$$\begin{aligned}
B = \mathcal{E}\,\frac{4}{9}[&8 + 5\,\pi^2 + 3\,\ln(\bar{\epsilon}_1) - 6\,\ln(2)\,\ln(\bar{\epsilon}_1) - 2\,\ln(\bar{\epsilon}_1)\,\ln(\bar{\epsilon}_2) + \ln(\bar{\epsilon}_2)^2 - 3\,\ln(\bar{\epsilon}_1') \\
&+ 6\,\ln(2)\,\ln(\bar{\epsilon}_1') + \ln(\bar{\epsilon}_1')^2]\,.
\end{aligned} \tag{29}$$

We can now go back to Eq. (24) for the second derivative of $\mathcal{E}_L$ which is a sum of A and B

$$\epsilon_2\,\frac{\partial\mathcal{E}_L}{\partial\epsilon_2} = \mathcal{E}\,\frac{4}{9}\,[8+\frac{3\,N}{2}+5\,\pi^2+3\,\ln(\bar\epsilon_1)-6\,\ln(2)\,\ln(\bar\epsilon_1)-2\,\ln(\bar\epsilon_1)\,\ln(\bar\epsilon_2)+\ln(\bar\epsilon_2)^2]\,. \tag{30}$$

The expression for $\mathcal{E}_L$ which matches both derivatives is:

$$\mathcal{E}_L(\bar\epsilon_1,\bar\epsilon_2) = \mathcal{E}\Big[\frac{2\,N\,\ln(\bar\epsilon_1)}{3}+\frac{32\,\ln(\bar\epsilon_2)}{9}+\frac{2\,N\,\ln(\bar\epsilon_2)}{3}+\frac{20\,\pi^2\,\ln(\bar\epsilon_2)}{9}+\frac{4\,\ln(\bar\epsilon_1)\,\ln(\bar\epsilon_2)}{3}$$
$$-\frac{8\,\ln(2)\,\ln(\bar\epsilon_1)\,\ln(\bar\epsilon_2)}{3}-\frac{4\,\ln(\bar\epsilon_1)\ln(\bar\epsilon_2)^2}{9}+\frac{4\ln(\bar\epsilon_2)^3}{27}\Big]\,. \tag{31}$$

The constant term (no logs) is not included here. E_L as shown in Fig. 1 is integrated over the region which is a combination of three rectangles:

$$E_L = \mathcal{E}_L\Big(\frac{\epsilon_1'}{\alpha^2},\frac{\epsilon_2}{\alpha^2}\Big)+\mathcal{E}_L\Big(\frac{\epsilon_2'}{\alpha^2},\frac{\epsilon_1}{\alpha^2}\Big)-\mathcal{E}_L\Big(\frac{\epsilon_1}{\alpha^2},\frac{\epsilon_2}{\alpha^2}\Big)\,. \tag{32}$$

IV. CONTRIBUTION E_M

In the one-loop case, contribution to energy, coming from photon energies $k^0 > \epsilon$ is

$$\delta E = \langle\phi|V|\phi\rangle\,, \tag{33}$$

$$V(\epsilon) = \alpha^2\,\delta^3(r)\Big[\frac{10}{9}-\frac{4}{3}\,\ln(2\,\epsilon)\Big]\,. \tag{34}$$

E_M is a V correction to the Bethe log:

$$E_M = \frac{2\,\alpha}{3\,\pi}\,\delta_{V(\epsilon_1)}\int_0^{\epsilon_2} d\omega\,\omega\,\langle\phi|p^i\,\frac{1}{E-(H+\omega)}\,p^i|\phi\rangle\,. \tag{35}$$

It has the same form as Eq. (23), so after symmetrization $\epsilon_1\leftrightarrow\epsilon_2$ it is:

$$E_M = \frac{\mathcal{E}}{2}\,\Big(\frac{10}{9}-\frac{4}{3}\,\ln(2\,\epsilon_1')\Big)\,\Big[-\frac{2}{3}\,\ln^2\frac{\epsilon_2}{\alpha^2}+2\,(1-2\,\ln 2)\,\ln\frac{\epsilon_2}{\alpha^2}+N\Big]+(\epsilon_1\leftrightarrow\epsilon_2)\,. \tag{36}$$

V. CONTRIBUTION E_F

E_F is the two-loop contribution with regularized Coulomb interaction and with both photon energies limited from below by ϵ. It is a sum of three terms

$$E_F = E_F^1+E_F^2+E_F^3\,, \tag{37}$$

defined and calculated as follows. E_F^1 is a second order correction coming from $V(\epsilon_1)$ and $V(\epsilon_2)$ with V defined in (34), here additionally with λ-regularization

$$E_F^1 = \langle\phi|V(\epsilon_1)\,\frac{1}{(E-H)'}\,V(\epsilon_2)|\phi\rangle\,. \tag{38}$$

The corresponding matrix element is given in Eq. (26), so E_F^1 becomes

$$E_F^1 = \frac{\mathcal{E}}{16}\left(\frac{10}{9}-\frac{4}{3}\,\ln(2\,\epsilon_1)\right)\left(\frac{10}{9}-\frac{4}{3}\,\ln(2\,\epsilon_2)\right)\left(-4\,\lambda-16\,\ln\frac{\lambda}{3}-4-64\,\ln\frac{3}{4}\right). \tag{39}$$

One needs only $\ln\lambda$ term, since others do not give $\ln\alpha$. E_F^2 is the contribution from electron formfactors F_1' and F_2 at $q^2=0$ on relativistic (Dirac) wave function. We know it from the one-loop case that for vacuum-polarization $A_{61}=A_{40}/2$. The same holds for two-loop contribution, thus we have

$$E_F^2 = \mathcal{E}\,\ln\alpha^{-2}\,\frac{B_{40}}{2}\,. \tag{40}$$

Diagrams with closed fermion loop are automatically included in the above formula. Other contributions coming from these diagrams are calculated in Section VII.

E_F^3 is the contribution from F_1'' and F_2' calculated with nonrelativistic wave functions. It leads to the matrix element $\langle\phi|\nabla^2\,\delta^3(r)_\lambda|\phi\rangle$ which does not lead to $\ln\lambda$. Hence, it does not contribute to $\ln\alpha$.

VI. CONTRIBUTION E_H

E_H is the contribution obtained from the two–loop three–photon exchange forward scattering amplitude. It requires subtractions of terms, contributing to Lamb shift at lower orders. After subtractions it is finite and depends on ϵ_1,ϵ_2 and $\Lambda=\lambda\,\alpha$. When combined with E_L and E_F, the dependence on ϵ_1,ϵ_2 and Λ should cancel out. Having this in mind, the $\ln\alpha$ contribution could be obtained by the replacement $\lambda\to 1/\alpha$ in E_F^1 in Eq. (39). However, the constant term B_{60} requires complete calculation of E_H, which we think is the most difficult of the contributions.

VII. DIAGRAMS WITH CLOSED FERMION LOOP

There is a small logarithmic contribution coming from diagrams with a closed fermion loop. They are partially included in E_F^2. Two other contributions E_{VP}^1, E_{VP}^2 are the following. The second order correction coming from the one-loop vacuum polarization is

$$E_{VP}^1 = \mathcal{E}\left(-\frac{4}{15}\right)^2\langle\phi|\delta^3_\lambda(r)\,\frac{1}{(E-H)}\,\delta^3_\lambda(r)|\phi\rangle \to \mathcal{E}\left(\frac{4}{15}\right)^2\,\ln\,\alpha\,. \tag{41}$$

The second contribution E_{VP}^2 is electron self–energy in the Coulomb potential including vacuum polarization correction. It is calculated in the similar way, as previous corrections. One splits it into three parts

$$E_{VP}^2 = C_L + C_M + C_H\,. \tag{42}$$

C_L is a v.p. correction $V=-(4/15)\,\delta^3(r)$ to the Bethe log:

$$
\begin{aligned}
C_L &= \frac{2\,\alpha}{3\,\pi}\,\delta_V \int_0^\epsilon d\omega\,\omega\,\langle\phi|p^i\,\frac{1}{E-(H+\omega)}\,p^i|\phi\rangle &(43)\\
&= \mathcal{E}\left(-\frac{4}{15}\right)\left(-\frac{2}{3}\,\ln^2\frac{\epsilon}{\alpha^2} + 2\,(1-2\,\ln 2)\,\ln\frac{\epsilon}{\alpha^2} + N\right). &(44)
\end{aligned}
$$

C_M is a second order correction coming from self–energy and v.p.

$$
\begin{aligned}
C_M &= 2\left(\frac{\alpha}{\pi}\right)^2\left(\frac{10}{9} - \frac{4}{3}\,\ln 2\,\epsilon\right)\left(-\frac{4}{15}\right)\langle\phi|\delta^3_\lambda(r)\,\frac{1}{(E-H)}\,\delta^3_\lambda(r)|\phi\rangle &(45)\\
&\to 2\,\mathcal{E}\left(\frac{10}{9} - \frac{4}{3}\,\ln 2\,\epsilon\right)\left(-\frac{4}{15}\right)\,\ln\alpha\,. &(46)
\end{aligned}
$$

C_H is given by the scattering amplitude. Since we calculate only the logarithmic part, instead of calculating B_H we replaced $\ln\lambda$ by $-\ln\alpha$ in the equation above. The logarithmic part of electron self–energy in the Coulomb potential including vacuum polarization correction is

$$
E^2_{VP} = \mathcal{E}\,\frac{4}{15}\left[\frac{2}{3}\,\ln^2\alpha^{-2} + 4\left(\frac{2}{9} + \ln 2\right)\,\ln\alpha^{-2}\right]. \tag{47}
$$

This completes the treatment of two-loop logarithmic correction

VIII. SUMMARY

The sum of all logarithmic terms in Eqs. (32,36,37,41,47) is

$$
\begin{aligned}
B_{63} &= -\frac{8}{27} &&= -0.296296\,, &(48)\\
B_{62} &= \frac{104}{135} - \frac{16\,\ln 2}{9} &&= -0.461891\,, &(49)\\
B_{61} &= \frac{39751}{10800} + \frac{4\,N}{3} + \frac{55\,\pi^2}{27} - \frac{616\,\ln 2}{135} + \frac{3\,\pi^2\,\ln 2}{4} + \frac{40\,\ln^2 2}{9} - \frac{9\,\zeta(3)}{8} &&&(50)\\
&= 50.309654\,.
\end{aligned}
$$

First of all the result for B_{61} is surprisingly large, and reverses the sign of the overall logarithmic contribution. B_{63} agrees with the result obtained first in [8]. However, as it was pointed out by Yerokhin [10], the loop-by-loop diagram is the source of additional terms, which were not accounted for in the calculation in [8]. An additional result of this work is the state dependence of B coefficients which is obtained from n-dependence of matrix elements in Eqs. (20,26,27)

$$
\begin{aligned}
B_{62}(n) &= B_{62} + \frac{16}{9}\left(\frac{3}{4} + \frac{1}{4\,n^2} - \frac{1}{n} - \ln(n) + \Psi(n) + C\right), &(51)\\
B_{61}(n) &= B_{61} + \frac{4}{3}\,(N(n) - N) + \left(\frac{304}{135} - \frac{32}{9}\,\ln(2)\right)\left(\frac{3}{4} + \frac{1}{4\,n^2} - \frac{1}{n} - \ln(n) + \Psi(n) + C\right). &(52)
\end{aligned}
$$

n-dependence of B_{62} agrees with the former result in [14] (apart from the misprint in the overall sign there). B_{61} depends on N-coefficient, the Dirac delta correction to Bethe logs, which has not been calculated yet for other states than 1S, therefore its complete state dependence is unknown. However, one may expect to a good approximation N is independent of n, as it is for Bethe logs.

Because of the large value of B_{61} theoretical predictions for hydrogen Lamb shift are going to be changed. The total logarithmic contribution is 16.9 kHz for the 1S state, compared to the previous one, based only on B_{63} -28.4 kHz. Theoretical predictions for Lamb shift in hydrogen with proton radius $r_p = 0.862(12)$ fm from [15], using recent updates: analytical calculations of the three-loop contribution by Melnikov and Ritbergen in [16] and direct numerical calculation of one-loop self-energy by Jentschura *et al.* in [17] are (see details in the appendix)

$$E_L(1S)_{\rm th} = 8\,172\,816(10)(32)\,{\rm kHz}, \tag{53}$$
$$E_L(2S-2P_{1/2})_{\rm th} = 1\,057\,842(1)(4)\,{\rm kHz}, \tag{54}$$

where we assumed for $B_{60} = 0\pm 100$, which gives the first uncertainty. For P-states we neglect B-terms completely. The second uncertainty comes from the proton charge radius. Since it dominates the theoretical error, we emphasize the importance of the muonic-hydrogen measurement, from which r_p could be precisely obtained. Current theoretical predictions agrees well with the most precise experimental values:

$$E_L(1S)_{\rm exp} = 8\,172\,837(22)\,{\rm kHz}\ [18,19], \tag{55}$$
$$E_L(2S-2P_{1/2})_{\rm exp} = 1\,057\,845(9)\,{\rm kHz}\ [20], \tag{56}$$
$$E_L(2S-2P_{1/2})_{\rm exp} = 1\,057\,842(12)\,{\rm kHz}\ [21]. \tag{57}$$
$$\tag{58}$$

Due to large uncertainty and ambiguities with the proton charge radius, one may regard the Lamb measurement as a determination of r_p. In this way, from 1S Lamb shift, one obtains:

$$r_p = 0.869(12)\,{\rm fm}\,. \tag{59}$$

Logarithmic two-loop corrections significantly alter theoretical predictions for the Lamb shift in the single ionized helium as well. The current theoretical value is

$$E_L(2S-2P_{1/2})_{\rm th} = 14\,041.57(8)\,{\rm MHz}\,. \tag{60}$$

It does not agree with both: the experimental value from [22] and the recent update in [23] respectively:

$$E_L(2S-2P_{1/2})_{\rm exp} = 14\,042.52(16)\,{\rm MHz}\,, \tag{61}$$
$$E_L(2S-2P_{1/2})_{\rm exp} = 14\,041.13(17)\,{\rm MHz}\,. \tag{62}$$

One may wonder about B_{60} and further higher order terms, keeping in mind the large value of B_{61}. There are two possible and complementary undergoing projects: direct calculation of this term or numerical calculation of complete two-loop diagrams with Dirac-Coulomb propagators. While the second would be the best way, the numerical accuracy might be limited at small Z, such as $Z = 1$. In the direct calculation of B_{60} one has to consider three points: two-loop Bethe logs with ϵ cut-offs, two-loop scattering amplitude with the photon mass μ, and the transition terms between ϵ and μ. This project seems to be achievable using the methods developed for B_{50}, positronium decay rate and the one applied here.

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APPENDIX A: FORMULAS FOR CALCULATIONS OF LAMB SHIFT

In the calculation of hydrogen and helium Lamb shift we use the following physical constants:

$$\begin{aligned} R &= 10973731.568516(84)\,\mathrm{m}^{-1}\,, \\ c &= 299792458\,\mathrm{m\,s}^{-}1\,, \\ \alpha^{-1} &= 137.03599958(50)\,, \\ \frac{m_p}{m_e} &= 1836.1526675(39)\,, \\ \frac{m_\alpha}{m_e} &= 7294.299508(16)\,, \\ r_p &= 0.862(12)\,\mathrm{fm}\,, \\ r_\alpha &= 1.673(1)\,\mathrm{fm}\,. \end{aligned} \tag{A1}$$

In general, Lamb shift in light hydrogen like systems is a sum of nonrecoil, recoil and the proton structure contributions. In the nonrecoil limit, known terms are:

$$\begin{aligned} E_L = m\,\frac{\alpha\,(Z\,\alpha)^4}{\pi\,n^3}\left(\frac{\mu}{m}\right)^3 &\{A_{40} + A_{41}\,L + (Z\,\alpha)\,A_{50} + (Z\,\alpha)^2\,[A_{62}\,L^2 + A_{61}\,L + A_{60}(Z\,\alpha)] \\ &+\frac{\alpha}{\pi}\,[B_{40} + (Z\,\alpha)\,B_{50} + (Z\,\alpha)^2\,(B_{63}\,L^3 + B_{62}\,L^2 + B_{61}\,L + B_{60}(Z\,\alpha))] \\ &+\left(\frac{\alpha}{\pi}\right)^2\,C_{40}\}\,, \end{aligned} \tag{A2}$$

where μ is a reduced mass, $m = m_e$, and $L = \ln[m/(\mu\,(Z\,\alpha)^2)]$. Most of these coefficients could be find in any review, such as [1] or [2]. The recent result is the direct numerical calculations of one-loop self-energy, which gives for hydrogen ($Z = 1$)

$$\begin{aligned} A_{60}(1S,\alpha) &= -30.29024 + \left[-0.6187 + \left(\frac{19}{45} - \frac{\pi^2}{27}\right)\right]\,, \\ A_{60}(2S,\alpha) &= -31.18515 + \left[-0.8089 + \left(\frac{19}{45} - \frac{\pi^2}{27}\right)\right]\,, \\ A_{60}(2P_{1/2},\alpha) &= -0.9735 - 0.0640\,, \end{aligned} \tag{A3}$$

and for He$^+$ ($Z = 2$)

$$\begin{aligned} A_{60}(2S,2\,\alpha) &= -30.64466 + \left[-0.7961 + \left(\frac{19}{45} - \frac{\pi^2}{27}\right)\right]\,, \\ A_{60}(2P_{1/2},2\,\alpha) &= -0.94940 - 0.0638\,, \end{aligned} \tag{A4}$$

where the second term is the vacuum polarization [24]. Another recent result is the analytical calculation of three-loop contribution in [16]. Together with the previously known vacuum polarization and anomalous magnetic moment it amounts to

$$C_{40} = 0.417508\,. \tag{A5}$$

In this work we calculate all logarithmic two-loop corrections for S-states. However, for P-state only B_{62} is known. For this reason in the theoretical predictions for hydrogen and helium we totally neglect higher order two loop corrections for P states, but included B_{40} only. We neglect also dependence of N in Eq. (20) on principal quantum number n, since N has not yet been calculated for $n \neq 1$. Recoil corrections, not included in Eq. (A2) sum to

$$\begin{aligned}\delta E = {} & \frac{\mu^3}{m\,M}\,\frac{(Z\,\alpha)^5}{\pi\,n^3}\left\{\frac{1}{3}\,\delta_{l0}\,\ln(Z\,\alpha)^{-2} - \frac{8}{3}\,\ln k_0(n,l) + \frac{14}{3}\,\delta_{l0}\left[\ln\left(\frac{2}{n}\right) + \Psi(n) + C + \frac{1}{2\,n} + 1\right]\right.\\ & \left. -\frac{1}{9}\,\delta_{l0} - \frac{2}{M^2-m^2}\,\delta_{l0}\left[M^2\,\ln\left(\frac{m}{\mu}\right) - m^2\,\ln\left(\frac{M}{\mu}\right)\right] - \frac{7}{3}\,\frac{1-\delta_{l0}}{l\,(l+1)\,(2\,l+1)}\right\}\\ & -\alpha\,\frac{(Z\,\alpha)^5}{n^3}\,\frac{m^2}{M}\,\delta_{l0}\,[1.364\,49(2)] + \frac{(Z\,\alpha)^6}{n^3}\,\frac{m^2}{M}\,D_{60}\,,\end{aligned} \tag{A6}$$

where

$$\begin{aligned}D_{60}(nS_{1/2}) &= 4\,\ln(2) - \frac{7}{2}\,,\\ D_{60}(l \geq 1) &= \left[3 - \frac{l\,(l+1)}{n^2}\right]\frac{2}{(4\,l^2-1)(2\,l+3)}\,.\end{aligned} \tag{A7}$$

The finite charge distribution of the nucleus and its self-energy give corrections:

$$\delta E = \frac{2}{3\,n^3}\,(Z\,\alpha)^4\,\mu^3\,r^2\,\delta_{l0} + \frac{4}{3\,\pi\,n^3}\,\frac{\mu^3}{M^2}\,(Z^2\,\alpha)\,(Z\,\alpha)^4\,\left[\ln\left(\frac{M}{\mu\,(Z\,\alpha)^2}\right)\delta_{l0} - \ln k_0(n,l)\right]\,. \tag{A8}$$

In the theoretical predictions, presented in this paper we have neglected higher order proton structure corrections and higher order recoil corrections, which at present are negligible.

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FIGURES

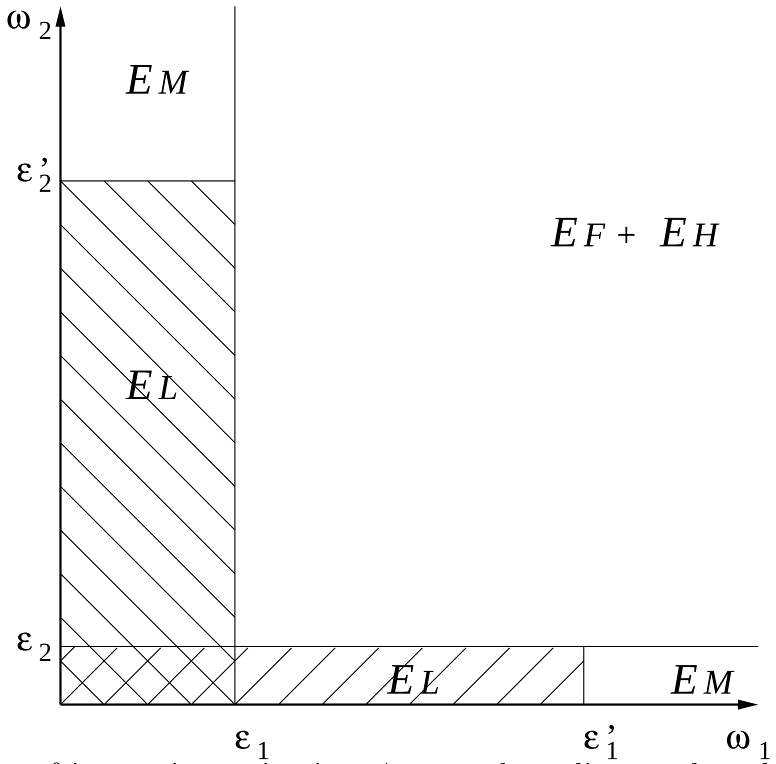

FIG. 1. division of integration region into 4 parts, depending on the value of both photon frequencies, $\epsilon_2 << \epsilon_1$

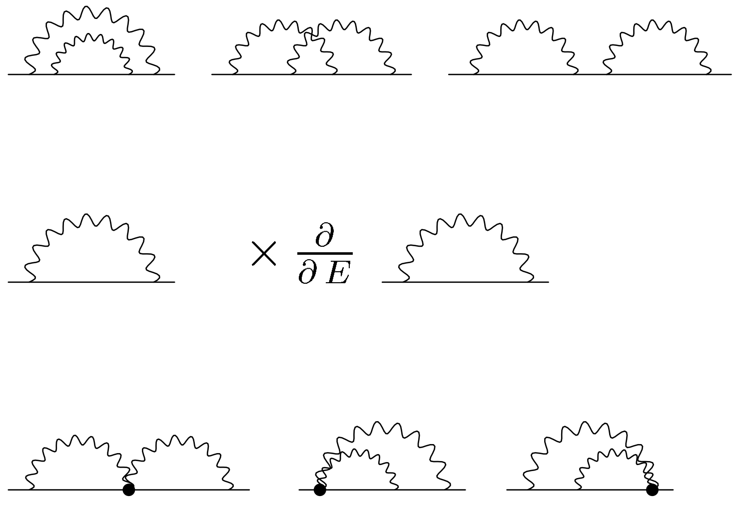

FIG. 2. Two–loop diagrams in the Coulomb gauge in NRQED