Determination of the Fine Structure Constant from Helium Spectroscopy

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Abstract. The fine structure of the 2^3P_J levels in helium is recalculated and the present status of theoretical predictions is summarized. Comparison with a recent high-accuracy measurement then allows the determination of the fine structure constant α. A detailed analysis of the higher-order terms which affect this determination is presented.

1. Introduction

The fine structure constant α can be obtained in a number of ways, each of which involves different energy scales and physical effects. A recent discussion of the various methods is given in Ref. [1]. Until recently the most accurate method involved assuming the validity of quantum electrodynamics (QED), applying it to the prediction of the electron anomalous magnetic moment a_e [2], and then extracting α by comparison with experiment [3]. The value determined is accurate to 3.8 parts per billion (ppb),

$$\alpha_{g-2}^{-1} = 137.035\ 999\ 58(52), \tag{1}$$

and is the most significant input to the 1998 CODATA [1] value of α,

$$\alpha_{\mathrm{CODATA}}^{-1} = 137.035\ 999\ 76(50). \tag{2}$$

A very promising alternative way to evaluate α is based on photon recoil effects in cesium [4]. The present result,

$$\alpha_{\mathrm{Cs-rec}}^{-1} = 137.035\ 999\ 710(427), \tag{3}$$

is accurate to 3.1 ppb, and has the potential of reaching the 1 ppb level. It is desirable to evaluate α in as many different systems as possible so as to check the internal consistency of QED. Because QED is a renormalizable quantum field theory, it should be equally applicable to all energy scales, although at higher energies it must be treated as part of the Standard Model. While no breakdown of the Standard Model has been observed up to several hundred GeV, QED tests involving α are of extremely high precision, and small deviations could be a sign of new physics. Of particular note in this regard is the recent measurement of the $g-2$ factor of the muon [5], where, assuming that hadronic effects are properly understood [6], a 2.6 standard deviation difference between theory and experiment has been observed. If this is attributed to the effect of a heavy particle, it will also affect the electron $g-2$, though at a much reduced level because of the smaller mass of the electron. For this reason, the $g-2$ of the electron may give a value of α slightly different from that determined from systems even less sensitive to high energy scales. To see this effect, precision measurements of α involving low energy scales, such as the cesium photon recoil experiment, are highly desirable.

One of the earliest high-accuracy determinations of α came from the fine structure of helium, the physics of which is characterized by the energy scale of a Rydberg, four orders of magnitude smaller than the electron mass, the relevant scale for the $g-2$ factor. In lowest order, fine structure is equal to the product of the Rydberg constant, the square of the fine structure constant, and factors that depend on the wave function of helium. To play a role in the present regime of high-accuracy determinations, both experiment and theory need to be brought to the 4 ppb precision level. Attempts to reach this precision for experiment are presently under way [7], and already the 15 ppb level has been reached by Hessels and collaborators [8]. Specifically, the so-called large splitting, $\nu_{01} = \nu(2^3P_0) - \nu(2^3P_1)$, has been measured to 30 ppb,

$$\nu_{01} = 29\ 616.950\ 9(9)\,\mathrm{MHz}, \tag{4}$$

which corresponds to a 15 ppb determination of α since it enters as a square. While not competitive for determining α, measurements of the small interval, $\nu_{12} = \nu(2^3P_1) - \nu(2^3P_2)$, provide a test of the QED calculations, and will also be considered in this paper. The two most accurate measurements of this interval are

$$\nu_{12} = \begin{cases} 2\ 291.174\ 0(14)\,\mathrm{MHz}\ [9] \\ 2\ 291.175\ 9(10)\,\mathrm{MHz}\ [10]. \end{cases} \tag{5}$$

It is the purpose of this paper to give a detailed description of the present status of the theory of helium fine structure and its attendant uncertainties. Two questions that clearly must be addressed are whether theory is at the same level as present experiment, 0.9 kHz, and whether it can reach the 0.24 kHz level, which corresponds to a 4 ppb determination of α if experimental accuracy reaches that level.

We have previously treated helium fine structure in Ref. [11]. That paper, however, was principally concerned with describing results from a calculation of the dominant terms of order α^5 Ry. Contributions of lower order were taken from a compilation provided by Drake [12], which however used the 1986 value of α. Here, we give a unified presentation of all contributions, including a precise recalculation of terms up to order α^4 Ry, and determine α to be

$$\alpha^{-1}_{\mathrm{he-fs}} = 137.035\ 982\ 7(21)(230), \tag{6}$$

where the first error is experimental and the second one is associated with uncalculated higher order corrections, which in terms of ν_{01} correspond to 10 kHz.

In the following sections we present the details of the theory in orders α^2, α^4, and α^5 Ry. All corrections of order α^3 Ry and some of higher order are accounted for by including the electron anomalous magnetic moment in the α^2 Ry result. Recoil effects are included completely through order α^3 Ry, and partial results in the next order of α give an estimate of the theoretical uncertainty. In the last section we discuss the issue of assigning theoretical error to the uncalculated higher order corrections and prospects for bringing theory to the 4 ppb level. In this paper we follow the convention of using ν for energy levels and splittings when they are to be expressed in MHz, and E for formulas in theoretical units.

2. Leading α^2 Ry fine structure

The fine structure in lowest order comes from the differing expectation values of the spin-dependent part of the Breit Hamiltonian $H_{\rm fs}$,

$$E(2^3P_J) = \langle 2^3P_J|H_{\rm fs}|2^3P_J\rangle, \tag{7}$$

for $J = 0, 1$, and 2. Here

$$\begin{aligned} H_{\rm fs} = {} & \frac{\alpha}{4\,m^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{r^3} - 3\,\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{r}\,\boldsymbol{\sigma}_2\cdot\boldsymbol{r}}{r^5}\right)(1+a_e)^2 \\ & + \frac{Z\,\alpha}{4\,m^2}\left[\frac{\boldsymbol{r}_1}{r_1^3}\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1 + \frac{\boldsymbol{r}_2}{r_2^3}\times\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_2\right](1+2\,a_e) \\ & + \frac{\alpha}{4\,m^2\,r^3}\Big\{\boldsymbol{r}\times\boldsymbol{p}_2\cdot[\boldsymbol{\sigma}_2(1+2\,a_e) + 2\,\boldsymbol{\sigma}_1(1+a_e)] \\ & \qquad - \boldsymbol{r}\times\boldsymbol{p}_1\cdot[\boldsymbol{\sigma}_1(1+2\,a_e) + 2\,\boldsymbol{\sigma}_2(1+a_e)]\Big\}, \end{aligned} \tag{8}$$

where $\boldsymbol{r}$ denotes $\boldsymbol{r}_1 - \boldsymbol{r}_2$ and we have included the effect of the electron anomalous magnetic moment a_e. The problem of accurately evaluating these expectation values has a lengthy history, beginning with the first calculations of Breit [13], which had an accuracy of a few percent. An accuracy of 1 ppm was achieved by Schwartz [14], and values accurate to under 1 ppb have been presented by Yan and Drake [15]. Quite recently, an approach to the nonrelativistic Schrödinger equation that results in extremely high accuracy has begun to be used: Korobov [16], for example, has applied it to the solution of the ground state energy accurate to 28 digits. We use this approach in all of the calculations presented here. The expectation values of operators involving spin is calculated using the formulas

$$\langle \boldsymbol{Q}\cdot\boldsymbol{s}\rangle_J = 2\,i\,\epsilon_{jki}\,\langle j|Q^k|i\rangle\,\{1/2, 1/4, -1/4\}\,, \tag{9}$$

$$\langle \boldsymbol{s}\cdot\hat{Q}\cdot\boldsymbol{s}\rangle_J = -\,2\,\langle j|Q^{ij}|i\rangle\,\{1/2, -1/4, 1/20\}\,, \tag{10}$$

for total angular momentum $J = 0, 1, 2$ respectively, with state $|i\rangle$ normalized to $\langle i|j\rangle = \delta^{ij}/3$ and $\boldsymbol{s} = (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)/2$. Q^i, Q^{ij} are vector and symmetric traceless tensor operators respectively. It is convenient to include the leading recoil correction by multiplying by a factor $(m_r/m_e)^3$, where m_r is the reduced mass for the electron-nucleus system, with the nucleus being an alpha particle. The net result for fine structure is

$$\nu(2^3P_0) = \left(\frac{m_r}{m_e}\right)^3 \alpha^2\,\mathrm{Ry}_\infty c\Big[\frac{\mathcal{E}_1}{2}(1+a_e)^2 + \frac{\mathcal{E}_2}{2}(1+2a_e) + \frac{\mathcal{E}_3}{2}(1+\frac{4}{3}a_e)\Big],$$
$$\nu(2^3P_1) = \left(\frac{m_r}{m_e}\right)^3 \alpha^2\,\mathrm{Ry}_\infty c\Big[-\frac{\mathcal{E}_1}{4}(1+a_e)^2 + \frac{\mathcal{E}_2}{4}(1+2a_e) + \frac{\mathcal{E}_3}{4}(1+\frac{4}{3}a_e)\Big],$$
$$\nu(2^3P_2) = \left(\frac{m_r}{m_e}\right)^3 \alpha^2\,\mathrm{Ry}_\infty c\Big[\frac{\mathcal{E}_1}{20}(1+a_e)^2 - \frac{\mathcal{E}_2}{4}(1+2a_e) - \frac{\mathcal{E}_3}{4}(1+\frac{4}{3}a_e)\Big]. \quad (11)$$

with $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ being the expectation values of the three terms in Eq. (8). We first present values for the constants $\mathcal{E}_i$ using wave functions obtained from a Hamiltonian that does not include the mass polarization term $\boldsymbol{p}_1 \cdot \boldsymbol{p}_2/m_\alpha$;

$$\begin{aligned}\mathcal{E}_{1\,\infty} &= 0.180\ 161\ 326\ 88(4),\\ \mathcal{E}_{2\,\infty} &= -\,0.277\ 273\ 659\ 36(4),\\ \mathcal{E}_{3\,\infty} &= 0.411\ 824\ 604\ 72(4).\end{aligned} \quad (12)$$

In the following we use the 1998 CODATA value of the Rydberg constant

$$\mathrm{Ry}_\infty c = 3.289\ 841\ 960\ 368(25)\,10^9\,\mathrm{MHz}, \quad (13)$$

the electron-alpha particle mass ratio

$$\frac{m_e}{m_\alpha} = 1.370\ 933\ 561\ 1(29)\,10^{-4}, \quad (14)$$

and the experimental value of the electron anomalous magnetic moment [3]

$$a_e = 0.001\ 159\ 652\ 188\ 4(43). \quad (15)$$

We note that in some treatments of helium fine structure the expansion of a_e in terms of α,

$$a_e = \frac{\alpha}{2\pi} - 0.328\ 479\left(\frac{\alpha}{\pi}\right)^2 + 1.181\left(\frac{\alpha}{\pi}\right)^3 \quad (16)$$

is used. To facilitate comparison with earlier work we give in Table 1 this breakdown for the fine structure arising from these leading terms up to order α^5 Ry. Terms of order α^6 Ry are not shown, as they contribute only at the level of one Hz. The first row is the non-recoil result, with the factor $m_r/m_e \to 1$. Note that there are recoil corrections in the second and higher rows discussed below.

3. Recoil corrections

While we have included the overall effect of the finite nuclear mass by including the third power of the ratio of the reduced mass to the electron mass, there are further recoil effects that must be included. The first one is the effect of the mass polarization term on the nonrelativistic wave function, which modifies the values for $\mathcal{E}_{1\infty}$, $\mathcal{E}_{2\infty}$, and $\mathcal{E}_{3\infty}$ to

$$\begin{aligned}\mathcal{E}_1 &= 0.180\ 220\ 618\ 64(4),\\ \mathcal{E}_2 &= -\,0.277\ 401\ 358\ 68(4),\\ \mathcal{E}_3 &= 0.411\ 999\ 963\ 60(4).\end{aligned} \quad (17)$$

In addition, there is a recoil correction to the Breit Hamiltonian,

$$H_{\rm rec} = \frac{Z\,\alpha}{2\,m_e\,m_\alpha}\left[\frac{\boldsymbol{r}_1}{r_1^3}\times(\boldsymbol{p_1}+\boldsymbol{p_2})+\frac{\boldsymbol{r}_2}{r_2^3}\times(\boldsymbol{p_1}+\boldsymbol{p_2})\right]\cdot(\boldsymbol{\sigma_1}+\boldsymbol{\sigma_2})\,(1+a_e) \quad (18)$$

which gives rise to the additional terms

$$\begin{aligned}
\nu_{\rm rec}(2^3P_0) &= \left(\frac{m_r}{m_e}\right)^3\frac{m_e}{m_\alpha}\,\alpha^2{\rm Ry}_\infty c\,\frac{\mathcal{E}_4}{2}(1+a_e),\\
\nu_{\rm rec}(2^3P_1) &= \left(\frac{m_r}{m_e}\right)^3\frac{m_e}{m_\alpha}\,\alpha^2{\rm Ry}_\infty c\,\frac{\mathcal{E}_4}{4}(1+a_e),\\
\nu_{\rm rec}(2^3P_2) &= \left(\frac{m_r}{m_e}\right)^3\frac{m_e}{m_\alpha}\,\alpha^2{\rm Ry}_\infty c\,\left(-\frac{\mathcal{E}_4}{4}\right)(1+a_e),
\end{aligned} \quad (19)$$

where

$$\mathcal{E}_4 = 0.241\ 945\ 125\ 64(4). \quad (20)$$

If mass polarization is not included then $\mathcal{E}_{4\,\infty} = 0.241\ 888\ 678\ 68(4)$. The second row of Table 1 gives the $m\,\alpha^5$ contribution with all recoil corrections included. Recoil corrections for the higher order terms in the Table 1 are negligible.

4. Higher order QED corrections

The above formulas have been written as a function of arbitrary α. The Rydberg constant, a_e, and the constants $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, and $\mathcal{E}_4$ are known with precision well under both the present experimental uncertainty and the desired goal of 4 ppb. To extract α from experiment, we now need to consider higher order QED corrections, starting in order α^4Ry. At that level we no longer need to leave α arbitrary, because uncertainties in its value lead to only negligible changes. Therefore we will use the 1998 CODATA value of α for all higher order terms. It is convenient to denote the higher order corrections to ν_{01} as $\delta\nu_{01}$, with

$$\delta\nu_{01} \equiv \nu_{01}^{(4)} + \nu_{01}^{(4r)} + \nu_{01}^{(5)} + \nu_{01}^{(6)}. \quad (21)$$

A similar breakdown will be given for corrections to the small interval, $\delta\nu_{12}$. Here we have separated the corrections into four classes. The first class, $\nu_{01}^{(4)}$, is the non-recoil limit of the α^4Ry contribution. It is known with high accuracy. The second, $\nu_{01}^{(4r)}$, accounts for recoil corrections at this order, that is, terms of order $m_e/m_\alpha\alpha^4$Ry. We will present partial results only for this term. The third, $\nu_{01}^{(5)}$, consists of all terms of order α^5Ry, which are again known only partially. Finally $\nu_{01}^{(6)}$, the correction of order α^6Ry, although nominally small, is enhanced by the presence of $\ln\alpha$ and so may be important at the 4ppb precision level; we will present rough estimates for it below. We are not aware of any further corrections which might play any role at this level of precision: for example, the nuclear finite size effect is negligible.

To calculate α from an experimental determination of ν_{01}, we need simply combine Eqns. (11) and (19) to find

$$\nu_{01} - \delta\nu_{01} = \left(\frac{m_r}{m_e}\right)^3\alpha^2\,{\rm Ry}_\infty c$$

$$\cdot \Big[\frac{3}{4}\,\mathcal{E}_1(1+a_e)^2 + \frac{\mathcal{E}_2}{4}\,(1+2\,a_e) + \frac{\mathcal{E}_3}{4}\,(1+\frac{4}{3}\,a_e) + \frac{m_e}{m_\alpha}\,\frac{\mathcal{E}_4}{4}(1+a_e)\Big]$$
$$= \alpha^2 \cdot 556\;200\;289.5(1)\mathrm{MHz}. \tag{22}$$

The present status of our calculation of higher order corrections to the large interval is

$$\delta\nu_{01} = -1.475(10)\mathrm{MHz}, \tag{23}$$

which then leads to the value of α given in Eq. (6). For the small interval we use the 1998 CODATA value of α in the lower order results to find

$$\nu_{12} = 2\;297.717\;8\,\mathrm{MHz} + \delta\nu_{12}, \tag{24}$$

which when taken together with the result of the present paper,

$$\delta\nu_{12} = -6.564(10)\,\mathrm{MHz}, \tag{25}$$

gives a result

$$\nu_{12} = 2\;291.154(10)\,\mathrm{MHz}\,, \tag{26}$$

in poor agreement with experiment. We now turn to a discussion of the four contributions to $\delta\nu_{01}$ and $\delta\nu_{12}$.

5. α^4 Ry Contributions

This contribution is a sum of a correction from the Breit interaction in second order perturbation theory and a correction coming from a set of operators $H^{(4)}_{\mathrm{fs}}$ that we refer to as Douglas-Kroll terms,

$$\begin{aligned}
E^{(4)}_{\mathrm{fs}} &= E^{(4)}_A + E^{(4)}_B\,,\\
E^{(4)}_A &= \langle 2^3P_J|H_{\mathrm{fs}}\,\frac{1}{(E-H)'}\,H_{\mathrm{fs}}\,|2^3P_J\rangle + 2\,\langle 2^3P_J|H_{\mathrm{nf}}\,\frac{1}{(E-H)'}\,H_{\mathrm{fs}}\,|2^3P_J\rangle,\\
E^{(4)}_B &= \langle 2^3P_J|H^{(4)}_{\mathrm{fs}}|2^3P_J\rangle.
\end{aligned} \tag{27}$$

The formula for $E^{(4)}_A$ involves the spin-independent part of the Breit Hamiltonian, the relevant part of which is

$$\begin{aligned}
H_{\mathrm{nf}} = &-\frac{1}{8\,m^3}\,(p_1^4+p_2^4) + \frac{Z\,\alpha\,\pi}{2\,m^2}\,[\delta^3(r_1)+\delta^3(r_2)]\\
&-\frac{\alpha}{2\,m^2}\,p_1^i\left(\frac{\delta^{ij}}{r}+\frac{r^i\,r^j}{r^3}\right)p_2^j.
\end{aligned} \tag{28}$$

The first complete derivation of the formulas for the α^4 Ry terms was given by Douglas and Kroll [17] in the framework of the Salpeter equation. A considerably simpler derivation of the same contributions is given in Ref. [18] using effective field theory.

The second order contribution involves intermediate states which have odd parity and spin-angular momentum ^3P, ^1P, ^3D, ^1D, and ^3F. We note that some authors include mass polarization along with the Breit Hamiltonian in this second order calculation, but that effect has already been accounted for here by evaluating the constants $\mathcal{E}_1, \mathcal{E}_2$, and $\mathcal{E}_3$ with helium wave functions that include mass polarization.

A complete tabulation of the numerous contributions that arise when $H_{\rm fs}$ and $H_{\rm nf}$ are split into their separate parts for the five intermediate states was given by Lewis and Serafino [19]. However, the singular nature of the Breit Hamiltonian makes accurate control of the infinite sum over states numerically difficult, and significant differences with Ref. [19] were found by Yan and Drake [15]. The result can be written as

$$\nu_A^{(4)} = \alpha^4\,\mathrm{Ry}_\infty c \sum_{i=1}^{5} A_i. \tag{29}$$

We tabulate results of our calculations for the contributions to the $J = 0, 1$, and 2 states in Table II. The first row for each J is the result with the anomalous magnetic moment included in the Breit Hamiltonian. The second row gives the result when it is omitted, for comparison with other calculations. Here we use only the first row to determine the contributions to the large and small intervals of

$$\nu_{01A}^{(4)} = 1.736\ 59(10)\ \mathrm{MHz}\,, \tag{30}$$

$$\nu_{12A}^{(4)} = -8.066\ 28(10)\ \mathrm{MHz}\,. \tag{31}$$

The second set of contributions $E_B^{(4)}$ come from operators intrinsically of order α^4 Ry. Some of these have already been included through our use of the experimental anomalous magnetic moment of the electron: they correspond to the term H_{16} in the compilation of Ref. [20], where the numerical values of the operators derived by Douglas and Kroll [17] were first presented, and have the numerical value tabulated in the third row of Table 1. We use the same ordering as Ref. [20], and write the remaining terms as

$$\nu_B^{(4)} = \alpha^4\,\mathrm{Ry}_\infty c \sum_{i=1}^{15} B_i, \tag{32}$$

and tabulate the constants B_i in Table 3. This leads to

$$\nu_{01B}^{(4)} = -3.283\ 65\,\mathrm{MHz}, \tag{33}$$

$$\nu_{12B}^{(4)} = 1.512\ 26\,\mathrm{MHz}. \tag{34}$$

Combining Eqns. (29) and (32) then gives the total

$$\nu_{01}^{(4)} = -1.547\ 06(10)\,\mathrm{MHz}, \tag{35}$$

$$\nu_{12}^{(4)} = -6.554\ 01(10)\,\mathrm{MHz}. \tag{36}$$

These results are in apparent disagreement with the α^4 results quoted in Table II of Ref. [15] for two reasons, both associated with the electron anomalous magnetic moment a_e. Firstly, we do not include H_{16}: however, this can be accounted for if the α^4 Ry terms coming from Eq. (8) associated with a_e, tabulated in Table 1, are added in. Secondly, despite the statement after Eq. (1) of Ref. [15] that the Breit Hamiltonian includes anomalous magnetic moment corrections, the results quoted for second order perturbation theory do not in fact include them. For comparison, then, the second row of Table 2 must be used. After these adjustments are made, the agreement between the two calculations is very good.

6. Recoil corrections of order $m_e/m_\alpha\,\alpha^4\,\mathrm{Ry}$

The calculations in the previous section were carried out with wave functions that did not include mass polarization, and with no factors of m_r/m_e. It is relatively simple to include both effects for the 15 Douglas- Kroll operators. Simple power counting indicates that a factor of $(m_r/m_e)^5$ should be multiplied into most of the contributions, but the second, third, fourth and fifth enter with a factor $(m_r/m_e)^4$. It is also straightforward to include mass polarization. These effects, tabulated in Table 3, sum to

$$\nu_{01B}^{(4r)} = \alpha^4\,\mathrm{Ry}_\infty c\ 0.000\ 190 = 0.001\ 78\,\mathrm{MHz}, \tag{37}$$

$$\nu_{12B}^{(4r)} = \alpha^4\,\mathrm{Ry}_\infty c\ 0.000\ 158 = 0.001\ 48\,\mathrm{MHz}. \tag{38}$$

It is noteworthy that while the effect of the m_r/m_e factors can be as large as 5 kHz for some terms, there is significant cancellation between them and the effect of mass polarization.

The above results are in good agreement with the result quoted for recoil effects in order α^4 Ry given in Drake's compilation [12]. However, the calculation is not complete. Firstly, the second order contribution $E_A^{(4)}$ also acquires reduced mass and mass polarization corrections. Secondly, the recoil correction to H_{fs}, H_{rec} (Eq. (18)), along with an analogous recoil correction to H_{nf}, must be included in the second order perturbation formula. Finally, a set of operators that generalize H_{rec} to order α^4 Ry must be derived and evaluated. While a set of operators has been given by Zhang [21], they have not been numerically evaluated or independently checked. Using the size of the largest terms actually calculated, 5 kHz, as an estimate of the possible size of the remaining uncalculated terms, we quote

$$\nu_{01}^{(4r)} = 0.001\ 78(500)\,\mathrm{MHz}, \tag{39}$$

$$\nu_{12}^{(4r)} = 0.001\ 48(500)\,\mathrm{MHz}. \tag{40}$$

7. α^5 Ry Contributions

Some constant terms in this order have already been included because we include the electron anomalous magnetic moment in H_{fs}. The remaining terms of this order can be represented as

$$E^{(5)} = \langle H^{(5)}\rangle + 2\left\langle H_{\mathrm{fs}}\,\frac{1}{(E-H)'}\,H^{(3)}\right\rangle + E_L, \tag{41}$$

where E_L is associated with low energies, and is related to Bethe logs. H_{fs} is now understood to have the anomalous magnetic moment correction dropped. $H^{(3)}$ is a spin independent effective Hamiltonian of order $m\,\alpha^5$ given by

$$H^{(3)} = -\frac{7\,\alpha^2}{6\,\pi\,m^2}\,\frac{1}{r^3}+\frac{Z\,\alpha^2}{m^2}\left[\delta^3(r_1)+\delta^3(r_2)\right]\left[\frac{4}{3}\,\ln(Z\,\alpha)^{-2}+\frac{10}{9}-\frac{4}{15}\right] \tag{42}$$

The logarithmic part of Eq. (41) was first derived by Zhang in [22], and later confirmed in [18]. Its numerical value was obtained by Yan *et al* in [23];

$$\delta\nu_{01} = \quad 0.082\ 59\,\mathrm{MHz}, \tag{43}$$

$$\delta\nu_{12} = -0.010\ 09\,\mathrm{MHz}. \tag{44}$$

The calculation of the constant term is much more complicated. We have obtained in Ref. [11] the complete value [24] for the second and third term in Eq. (41), and a partial value for the first one. They contribute to the large and small intervals the amount of

$$\delta\nu_{01} = -0.012\ 30(16)\,\mathrm{MHz}, \tag{45}$$

$$\delta\nu_{12} = -0.001\ 58(27)\,\mathrm{MHz}. \tag{46}$$

$H^{(5)}$ consists of very large numbers of quite singular operators. Many of them have been obtained by Zhang in [22] using a Salpeter formalism. We have noted in Ref. [11] several inconsistencies with our results and therefore we claim that the complete $H^{(5)}$ has not yet been derived. For these reasons, we have to associate some uncertainty to the uncalculated part of $H^{(5)}$. We note that two contributions of this order are known, which have already been included in our treatment. The first comes from the effect of the anomalous magnetic moment of the electron on the lowest order fine structure formula, tabulated in Table 1, which contributed 0.6 kHz. The second is from its effect on second order perturbation theory, where it contributed 8.9 kHz, as can be seen from Table 2. As will be discussed below, there are a very large number of uncalculated terms from electron-electron interactions. However, the already calculated terms above from electron-nucleus interactions tend to be larger, so we assign an overall uncertainty of 5 kHz. The presently known result of order α^5 Ry is a sum of Eqs. (43,44) and Eqs. (45,46) with this uncertainty

$$\nu_{01}^{(5)} = 0.070\ 29(500)\,\mathrm{MHz}, \tag{47}$$

$$\nu_{12}^{(5)} = -0.011\ 67(500)\,\mathrm{MHz}. \tag{48}$$

8. α^6 Ry Contributions

A major concern, given the fact that even the $m\alpha^7$ calculation is so difficult that to date no complete evaluation has appeared, is the question of the size of $m\alpha^8$ terms. If these enter at greater than the 4 ppb level (0.24 kHz), it will be difficult to use helium fine structure to determine α. For this reason we investigated the dominating logarithmic terms. While $\ln^2\alpha$ terms could be expected, we have found only $\ln\alpha$ terms. One source of such terms has recently been identified [25] in hydrogen fine structure. This comes from the two–loop self–energy contribution, which for the hydrogen-like case is

$$\Delta E = -\frac{m}{3}\left(\frac{\alpha}{\pi}\right)^2 Z^6\,\alpha^6\,\ln(Z\,\alpha)^{-2}\,\frac{1}{n^2}\left(1-\frac{1}{n^2}\right), \tag{49}$$

where n is the principal quantum number. The dependence on n comes from the following operator

$$\frac{Z^5}{n^3}\left(1-\frac{1}{n^2}\right) = -i\,\pi\,\langle \boldsymbol{p}\times\delta^3(r)\,\boldsymbol{p}\cdot\boldsymbol{\sigma}\rangle. \tag{50}$$

This allows one to write analogous contribution for helium fine structure as

$$\Delta E_1 = \frac{Z}{3\,\pi}\,m\,\alpha^6\,\ln(Z\,\alpha)^{-2}\,i\,\langle \boldsymbol{p}_1\times\delta^3(r_1)\,\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1 + \boldsymbol{p}_2\times\delta^3(r_2)\,\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_2\rangle. \tag{51}$$

Another logarithmic contribution at this order is implicit in second term in Eq. (41). If we denote the anomalous magnetic moment correction in $H_{\rm fs}$ by $H_{\rm fs}^{\rm an}$, this contribution is then

$$\Delta E_2 = \frac{8\,Z\,\alpha^2}{3\,m^2}\,\ln(Z\,\alpha)^{-2}\,\left\langle H_{\rm fs}^{\rm an}\,\frac{1}{(E-H)'}\,[\delta^3(r_1)+\delta^3(r_2)]\right\rangle. \tag{52}$$

The one-loop radiative correction does not lead to logarithmic electron-nucleus terms. Therefore further logarithmic contributions come from electron-electron interaction terms, which should be smaller than the terms derived here. Numerically, they contribute

$$\begin{aligned} \nu_{01}^{(5)} &= \quad 0.000\;08(30)\ \mathrm{MHz}, \\ \nu_{12}^{(5)} &= -\,0.000\;14(30)\ \mathrm{MHz}. \end{aligned} \tag{53}$$

where the uncertainty of .3 kHz is a rough estimate of the constant coefficient. Because these values are much smaller than our estimated total uncertainty, we have not included them in final results.

9. Summary

We have presented a complete set of QED contributions to helium fine structure, and recalculated most of them with high accuracy. The current precision of the theoretical result is 10 kHz: 5 kHz come from the α^5 Ry contribution and 5 kHz from recoil correction to the α^4 Ry contribution. Therefore, the answer to the question of whether theory is as precise as experiment is negative: at present, the determination of α given in Eq. (22) is not competitive with respect to other methods.

With regard to the question of whether theory has the possibility of reaching the 4 ppb level, we are more optimistic. If terms of order α^6 Ry were important, this would not be the case, but we believe that we have identified the largest logarithmic contributions of that order, and they are an order of magnitude smaller than present experimental precision. Therefore, if the calculations of $\nu^{(4r)}$ and $\nu^{(5)}$ can be completed, and if experimental accuracy continues to improve, helium fine structure should provide a competitive test of the fine structure constant at energy scales of a Rydberg.

Of the two uncompleted terms, it seems to us that the complete calculation of the recoil correction is achievable with present methods. However, completion of the α^5 Ry contribution presents a real challenge. We have attempted, so far without success, to derive a complete set of operators in $H^{(5)}$. Of particular concern is that many of them are singular and require regularization to keep their matrix elements finite. Some of them are only conditionally convergent. However, the first task should be a complete derivation of all the relevant operators, particularly the most difficult set coming from the one–loop self–energy. To avoid mistakes, these operators should ideally be derived independently by two groups in a systematic manner, so results could be compared term by term. Once this has been done and the operators evaluated numerically, helium fine structure should play an important role in the determination of the fine structure constant and testing the internal consistency of QED.

Acknowledgments

The work of K.P. was supported by the Polish Committee for Scientific Research under contract No. 2P03B 057 18. That of J. Sapirstein was supported by NSF grant PHY-9870017 and NIST precision measurement grant 60NANB6D0036. Useful conversations with Eric Hessels are gratefully acknowledged, and we thank Gordon Drake for providing us with details of his calculations.

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Table 1. Electron anomalous magnetic moment contributions to helium fine structure in powers of α: units MHz

Order of α	2^3P_0	2^3P_1	2^3P_2	ν_{01}	ν_{12}
α^3Ry $(\frac{m_e}{m_\alpha}=0)$	36.102 421	-18.605 435	3.942 777	54.707 856	-22.548 212
α^3Ry	36.100 813	-18.603 226	3.941 773	54.704 039	-22.544 999
α^4Ry	-0.033 805	0.017 746	-0.003 887	-0.051 551	0.021 633
α^5Ry	0.000 395	-0.000 205	0.000 044	0.000 600	-0.000 248

Table 2. Coefficients A_i for $\nu_A^{(4)}$: for each J, first row with, second row without anomalous magnetic moment included in Breit Hamiltonian.

State	^3P	^1P	^3D	^1D	^3F
$J=0$	-0.653470(9)	0.0	0.0	0.0	0.0
	-0.651602(9)	0.0	0.0	0.0	0.0
$J=1$	-0.127293(5)	-0.709495(9)	-0.002831	0.0	0.0
	-0.126969(5)	-0.707000(9)	-0.002823	0.0	0.0
$J=2$	0.041297(5)	0.0	-0.008262	-0.002385	-0.005626(1)
	0.041278(5)	0.0	-0.008236	-0.002379	-0.005600(1)

Table 3. Douglas-Kroll terms in the infinite nuclear mass limit $\nu_B^{(4)}$, reduced mass and mass polarization corrections $\nu_B^{(4r)}$, in units of α^4 Ry = 9.329027 MHz.

contribution	$\nu_{01B}^{(4)}$	$\nu_{12B}^{(4)}$	$\nu_{01B}^{(4r)}$	$\nu_{12B}^{(4r)}$
B_1	0.224 210	0.448 420	0.000 081	0.000 162
B_2	0.016 418	0.032 836	-0.000 013	-0.000 026
B_3	0.010 203	-0.004 081	-0.000 045	0.000 018
B_4	0.034 557	0.069 115	-0.000 002	-0.000 003
B_5	0.081 157	-0.032 463	-0.000 009	0.000 003
B_6	-0.093 084	-0.186 167	-0.000 009	-0.000 018
B_7	-0.171 228	-0.342 456	0.000 054	0.000 108
B_8	0.000 221	0.000 442	0.000 007	0.000 015
B_9	0.006 456	0.012 912	-0.000 022	-0.000 044
B_{10}	-0.008 671	-0.017 343	0.000 001	0.000 002
B_{11}	-0.004 457	0.001 783	0.000 004	-0.000 001
B_{12}	-0.010 325	0.004 130	0.000 000	0.000 000
B_{13}	-0.786 253	0.314 501	0.000 235	-0.000 094
B_{14}	0.135 711	-0.054 284	-0.000 032	0.000 013
B_{15}	0.213 103	-0.085 241	-0.000 061	0.000 024
Total	-0.351 982	0.162 103	0.000 190	0.000 158