Quantum Electrodynamics of weakly bound systems

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An overview of quantum electrodynamic effects in two–body systems is presented. Recent advances in the calculation of the hydrogen Lamb shift, muonium hyperfine structure, and positronium energy levels are described in detail. The comparison of experimental results to current theoretical predictions gives agreement in most cases. However, a few significant discrepancies remain, which indicates the necessity for further refined calculations and measurements.
Keywords: quantum electrodynamics, high precision calculations, hydrogen, Lamb shift, fine and hyperfine structure, exotic atoms.

1. Introduction

Quantum Electrodynamics(QED) aims to give an accurate description of all physical phenomena connected with the electromagnetic force from macroscopic scales down to the subatomic scale. The effects of other interactions can in general be treated as small perturbations. However, the incompletely understood strong interaction leads to a limitation of the interpretation of QED for the fine and hyperfine structure of hydrogenic systems. This is because the distribution of charge and magnetic moment inside the proton or nucleus can not be precisely determined.

Quantum electrodynamics has not yet been formulated in a mathematically strict manner. It exists as a compound of prescriptions of how to perturbatively calculate physical quantities such as cross sections or bound state energies. It is understood that all the ultraviolet divergences appearing in multi-loop integrations can be absorbed into the redefinition of the mass and the charge of the electron. This renormalisation theory, which has been proved to work in all orders of perturbation theory, has been applied in practice through eighth order in e.

The first decisive confirmation of QED came from the measurement of the $2S_{1/2}-2P_{1/2}$ splitting in the hydrogen atom [1]. According to Dirac equation both levels $2S_{1/2}$ and $2P_{1/2}$, while shifted by relativistic effects, remain degenerate. The electron self–energy and vacuum–polarization effects split these states by an amount in good agreement with the measured value. Currently, the Lamb shift in hydrogen has been measured very precisely $\delta E/E \approx 10^{-6}$ by several groups, which serves to rule out many exotic theories.

Many other bound state QED tests have been performed since the original Lamb shift measurement. The most precise are the $2S-4S$ two–photon transition in hydrogen by the Hänsch group in Garching [2], the $2S-8S$ transition by Biraben *et al.* in Paris [3], hydrogen [4] and muonium [5] hyperfine structure, the hyperfine splitting [6] and the $1S-2S$ transition [7] in positronium, and finally the $2S_{1/2}-2P_{1/2}$ transition in singly ionized helium. The last transition, which was measured by Wijngaarden, *et al.* in Windsor [8], is in disagreement by 8 standard deviations with the theoretical predictions [9]. This discrepancy persists in spite of the fact that for a very similar system, namely hydrogen, theoretical predictions agree well with all the measurements. The last example we should mention here is a long standing discrepancy for the orthopositronium decay rate. Although there is some inconsistency between different experiments [10], there is an indication of disagreement with the theoretical predictions [11]. However, the very difficult two–loop calculation has not been completed, and if that result is sufficiently large, agreement between theory and experiment may be restored.

In this short review I present a treatment of the Lamb shift and the hyperfine structure for hydrogen, muonium and positronium. Moreover, state of the art QED calculations are briefly described to give the reader a flavor of the recent advances in QED theory.

2. Hydrogen Lamb shift

Hydrogen is the most intensively studied bound state system, both experimentally and theoretically. While progress in high precision hydrogen spectroscopy is described by Biraben in his contribution to this Conference, I concentrate mostly on the theoretical description. Usually, as a first approximation one assumes that the proton is a static source of a Coulomb field, and later considers so called recoil corrections: for details see [12]. QED predicts electron self–interactions that leads to the shift of its energy. The derivation of this effect involves the perturbative expansion in the number of "loops", i.e. number of photons involved in the self–interaction. One derives the following complete relativistic expression for the one–loop self–energy in the field of a nucleus of charge $Z\,e$:

$$\Delta E = e^2 \int \frac{d^4k}{(2\pi)^4 i}\, g_{\mu\nu} \left(\frac{1}{k^2} - \frac{1}{k^2+\Lambda^2}\right) \times \langle\overline{\psi}|\gamma^\mu \frac{1}{\not p - \not k + \frac{Z\,\alpha}{r}\gamma^0 - m}\gamma^\nu|\psi\rangle - \langle\overline{\psi}|\delta m(\Lambda)|\psi\rangle\,, \tag{1}$$

which is represented by the Feynman graph of Fig. (1). Here p_0 is the bound

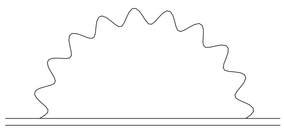

Figure 1. Electron self-energy diagram

state energy, and δm is a mass counter term. It is conventional to keep the Z dependence in the above even for our case, as it distinguishes radiative from binding corrections. The expression in Eq. (1) contains ultraviolet divergences which are characteristic of the perturbative formulation of QED. The integral is divergent in the limit $\Lambda \to \infty$. This divergence is canceled out by the δm term, using the requirement that for a free electron the energy shift, or in other words the correction to the electron mass, is exactly zero. The evaluation of the integral in Eq. (1) is a highly nontrivial problem. Two approaches have been introduced. The first, which has been applied with highest accuracy by P. Mohr [13], relies on a partial wave decomposition of the electron propagator and numerical integration. The other approach is based on an analytic expansion in $Z\,\alpha$ that is particularly suited for hydrogen [14], [15]. The original calculation of the leading term by Bethe [16] and later by Feynman and others explored the fact that $Z\,\alpha$ is small. To demonstrate specific features of the bound state calculation let me describe the derivation of the leading term. There are two characteristic energies in Eq. (1): the bound state energy $m\,\alpha^2/2$ and the electron mass m. One splits this integral into two parts and performs approximations specific to each part. In the low energy part the photon energy is limited by some arbitrarily chosen parameter ϵ that lies somewhere between 0 and the electron mass m. In the high energy part ϵ plays a role of an infrared cut-off. The key ingredient of this calculation is the fact that after expansion in $Z\,\alpha$ one goes to the limit of $\epsilon \rightarrow 0$. It is very helpful in the evaluation of the low energy part, where the binding field enters in a nonperturbative way. One can perform a non relativistic expansion of Eq. (1), use the Coulomb gauge, and find that the leading terms take the form

$$E_L = e^2 \int^\epsilon \frac{d^3k}{(2\,\pi)^3\,2\,k} \left(\delta^{ij} - \frac{k^i\,k^j}{k^2}\right) \langle\phi|p^i \frac{1}{E-\omega-H}\,p^j|\phi\rangle \tag{2}$$

$$= m\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^4}{n^3}\,\frac{4}{3}\ln\left(\frac{2\,\epsilon}{m\,(Z\,\alpha)^2\,k_0(n,l)}\right)\,. \tag{3}$$

It is understood that in the expression (2) the limit $\epsilon \rightarrow 0$ is performed after the expansion in $Z/,\alpha$. For the high energy part, E_H, one makes an expansion of the electron propagator in Eq. (1) in powers of Coulomb potential $V = -Z\,\alpha/r$. It is sufficient, for the leading order, to consider only terms up to V^2. A peculiar feature of the QED bound state calculations is the internal cancellation between terms containing different powers of V. In our case it is sufficient to calculate only the single V term with the external legs on mass shell, i.e. we need only two electromagnetic formfactors, F_1 and F_2. Expression for these can be found in any textbook on quantum field theory. The effective interaction (radiatively corrected) of an electron with the electromagnetic field takes the form

$$\gamma_\rho\,A^\rho \rightarrow \left(\gamma_\rho\left\{1+\frac{\alpha\,q^2}{3\,\pi\,m^2}\left[\ln\left(\frac{m}{2\,\epsilon}\right)+\frac{11}{24}\right]\right\}+\frac{i}{2\,m}\,\frac{\alpha}{2\,\pi}\,\sigma_{\rho\nu}\,q^\nu\right)\,A^\rho(q)\,. \tag{4}$$

The correction to the energy E_H is given by the expectation value of γA on the electron state,

$$E_H = e\,\langle\overline{\psi}|\gamma A|\psi\rangle = m\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^4}{n^3}\,\left[\frac{4}{3}\,\ln\left(\frac{m}{2\,\epsilon}\right)+\frac{11}{18}\,\delta_{l0}-\frac{1}{2\,\kappa\,(2\,l+1)}\right)\,, \tag{5}$$

where

$$\kappa = \begin{cases} -(l+1) & \text{for } j = l+\frac{1}{2} \\ \\ l & \text{for } j = l-\frac{1}{2} \end{cases} \tag{6}$$

The leading order one–loop self–energy contribution to the Lamb shift is a sum of E_L and E_H. However, it has been calculated in the non recoil limit, i.e. with the assumption that proton is a static source. Recoil corrections, in leading order, can be incorporated by a simple mass rescaling: for details see [12] and [17]. The complete formula which is valid for any mass ratio takes the form

$$\Delta E = m\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^4}{n^3}\left\{\left[\left(\frac{4}{3}\,\ln\frac{m}{\mu\,(Z\,\alpha)^2}+\frac{10}{9}\right)\,\delta_{l0}-\frac{4}{3}\,\ln k_0(n,l)\right]\,\frac{\mu^3}{m^3} -\frac{1}{2\,\kappa\,(2\,l+1)}\,(1-\delta_{l0})\,\frac{\mu^2}{m^2}\right\}. \tag{7}$$

The next contribution. which is presented here without a derivation, is associated with vacuum polarization. The Coulomb field of the nucleus is modified at distances of order $1/m$, the electron Compton wavelength, by electron–positron pair creation, what leads to the so called Ühling potential. In leading order the energy shift is given by

$$\Delta E_{VP} = \frac{\mu^3}{m^2}\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^4}{n^3}\left(-\frac{4}{15}\right) \tag{8}$$

These leading order terms are not sufficient to compare with the measurements. There are higher order terms $O(Z\,\alpha)$ and $o(Z\,\alpha)^2$ which have been calculated over a period of 5 decades, and for recent results I refer the reader to reviews in [12] and [9]. Let us concentrate on the $O(Z\,\alpha)$ correction, since its evaluation demonstrates further interesting features of bound state QED. It is a general feature for all bound states that the $\alpha\,(Z\,\alpha)^5$ contribution is given by the radiative correction to the two–photon exchange forward amplitude. In the case of hydrogen the corresponding diagrams are presented in Fig. (2). In this

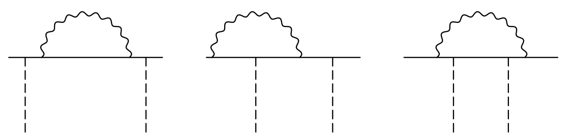

Figure 2. Radiative correction of order $\alpha\,(Z\,\alpha)^5\,m$ to the Lamb shift. The wavy line denotes a photon propagator and the dashed line a Coulomb interaction.

scattering amplitude, f, the external legs have momentum $p = (m, 0, 0, 0)$ so it depends only on the three dimensional momentum exchange q. The contribution to the energy has the form:

$$\Delta E = \frac{\mu^3}{m^2}\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^5}{n^3}\int\frac{d^3q}{(2\,\pi)^3}\frac{f(q^2)}{q^4}\,. \tag{9}$$

At low momentum the amplitude f behaves as $\approx [10/9 - 4/3\ln(q^2) + o(q^2)]$. These terms should be subtracted out, since they contribute to the lower order, which has already been calculated. The integral with the subtracted amplitude is finite, and amounts to:

$$\Delta E = \frac{\mu^3}{m^2}\,\frac{\alpha\,(Z\,\alpha)^5}{n^3}\,\left(\frac{139}{32} - 2\,\ln(2) + \frac{5}{48}\right)\,, \tag{10}$$

where the last term 5/48 comes from vacuum polarization. In the same treatment one accounts for the additional radiative recoil corrections that have not been accounted for by the reduced mass rescaling of the wave function at the origin [17]. One writes a scattering amplitude for two photon exchange between the electron and nucleus, subtracts lower order terms, and obtains the correction

$$\Delta E = \frac{m^2}{M}\,\alpha\,\frac{(Z\,\alpha)^5}{n^3}\,(-1.364\,49)\,. \tag{11}$$

There are many other contributions to the Lamb shift which have been worked out. The most important are the two–loop contribution and pure recoil corrections. At the time this article is being written, the two–loop contribution has been calculated only by means of expansion in $Z\,\alpha$ [18–20],

$$\Delta E = m\,\left(\frac{\alpha}{\pi}\right)^2\,\frac{(Z\,\alpha)^4}{n^3}\,\left[0.867\,420\,\delta_{l0} + 0.328\,478\,\frac{1}{\kappa(2\,l+1)} \right.$$
$$\left. -(Z\,\alpha)\,21.558(3)\,\delta_{l0} + \cdots\right]. \tag{12}$$

The unexpectedly large value of the $(Z\,\alpha)$ coefficient for S-states causes the theoretical prediction for the Lamb shift in hydrogen and singly ionized helium to shift beyond previous uncertainty estimates.

Pure recoil corrections may be understood as a correction to the Dirac equation due to motion of the nucleus. They require a much more complicated analysis involving the Bethe–Salpeter equation. This equation is frequently made more tractable by, for example, elimination of the relative time from the generalized wave function. Let me mention here that there exist alternative formalisms, for example: NRQED by P. Lepage [21] or a similar effective Hamiltonian method which was developed by me in [22]. They take advantage of the natural cancellation of many terms before performing the actual calculation. These formalisms are particularly suited for systems consisting of particles with comparable masses.

Coming back to hydrogen, the Bethe Salpeter equation was used by Salpeter [23] for the derivation of $(Z\,\alpha)^5$ contributions. Later on, Shabayev [24] found beautiful formulas for the recoil correction in first order of the mass ratio that are valid to all orders in $(Z\,\alpha)$. These formulas have been rederived in a much simpler way in [25]. They were the basis of the calculation of recoil effects in order $(Z\,\alpha)^6$ which I have performed with H. Grotch [25].

There are further corrections to the Lamb shift due to the finite size and polarizability of the nucleus. I postpone the detailed description of these effects to the next section on the hydrogen–deuterium isotope shift. Here we need only to know the leading finite size effect

$$\Delta E = \frac{4}{3}\, m^3\, (Z\,\alpha)^4\, \langle r_p^2\rangle \tag{13}$$

which is proportional to the square of the proton charge radius. The $(Z\,\alpha)$ corrections to this are roughly proportional to $\langle r_p^3\rangle$ and thus completely negligible for the case of the hydrogen Lamb shift. There is a serious problem with the proton charge radius. There are two measurements that disagree [26], [27] and each one has a large uncertainty. Moreover, even a new reanalysis [28] of the former scattering data is in disagreement with both previous values. We therefore stress the need of a new independent determination of the proton charge radius. In fact, there are plans at the Paul-Scherrer Institute in Villingen to measure the Lamb shift in muonic hydrogen. If we replace a hydrogenic electron by a muon, which is about 200 times heavier, then the value of the wave function at the origin is increased 200^3 times. This makes the system more sensitive to vacuum polarization, but also to nuclear size effects. For example, the finite size effect amounts to $\Delta E(2P - 2S) = -3.862(108)$ meV, compared to the Lamb shift $E_L(2P - 2S) = 205.932(10)$, which was calculated without $\Delta E(2P - 2S)$. In Ref. [29] I have recalculated all the contributions to the Lamb shift in muonic hydrogen. This would allow for a tenfold improvement of the proton charge radius, provided, of course, a Lamb shift measurement in μH is performed.

For the comparison of theoretical predictions with experimental results for the Lamb shift in hydrogen we assume the proton charge radius [27] $r_p = 0.862(12)$ fm. The theoretical predictions are

$$L(2S - 2P) = 1057.839(6)\,\mathrm{MHz} \tag{14}$$

$$L(4S - 2S) - \frac{1}{4}L(2S - 1S) = 868.622(5)\,\mathrm{MHz} \tag{15}$$

$$L(1S) = 8172.802(40)\,\mathrm{MHz} \tag{16}$$

in moderate agreement with several independent measurements

$$L(2S-2P) = 1057.845(9)\,\mathrm{MHz}\ [30] \tag{17}$$

$$L(2S-2P) = 1057.842(12)\,\mathrm{MHz}\ [31] \tag{18}$$

$$L(4S-2S) - \frac{1}{4}L(2S-1S) = 868.631(10)\,\mathrm{MHz}\ [2] \tag{19}$$

$$L(1S) = 8172.876(29)\,\mathrm{MHz}\ [32,3]\,, \tag{20}$$

where $L(1S)$ was obtained by comparing 1S-2S with 2S-8S. The various results are summarized in Fig. (3). However, there is a significant discrepancy for singly ionized helium He^+, which is a hydrogen–like system:

$$L_{th}(2S-2P) = 14041.33(18)\,\mathrm{MHz}\ [9] \tag{21}$$

$$L_{exp}(2S-2P) = 14042.52(16)\,\mathrm{MHz}\ [8] \tag{22}$$

$$\Delta = L_{th} - L_{exp} = 1.19(18)(16)\,\mathrm{MHz}\,. \tag{23}$$

All the corrections obtained for hydrogen require only a simple rescaling due to the different properties of the nucleus, specifically the charge, mass, and charge

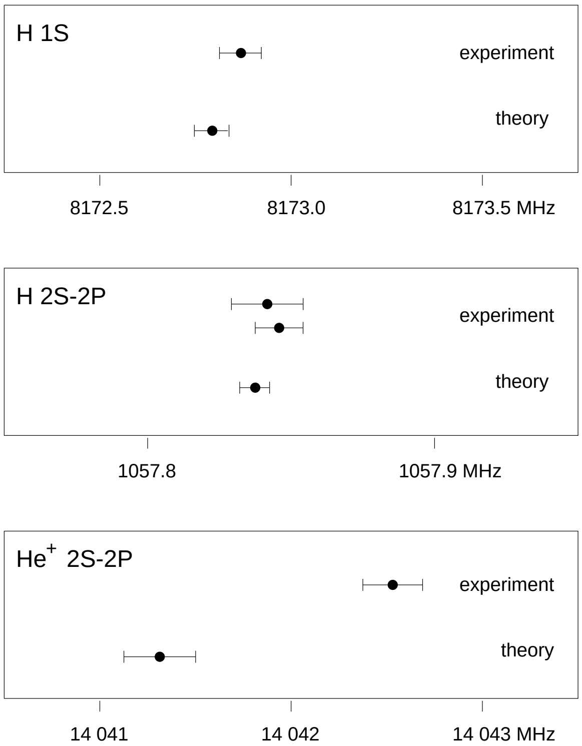

Figure 3. Lamb shift in H and He^+

radius. The fact that the helium nucleus has spin 0 rather than 1/2 results in shifts of only a few kHz, which is not presently significant. A leading higher order effect of order $m\,\alpha^2\,(Z\,\alpha)^6\ln^3(Z\alpha)^{-2}$ has been calculated by Karshenboim [33]. It would shift the theoretical prediction for the 2S-2P Lamb shift by -3.6 kHz, and -146 kHz in He^+, which is of the opposite sign and insufficient magnitude to explain the discrepancy in this ion. If the measurement is confirmed, this will indicate a serious disagreement with quantum electrodynamic theory.

3. Hydrogen–Deuterium isotope shift and the structure of the Deuteron

The precision spectroscopy of the $1S-2S$ line in hydrogen and deuterium has reached an accuracy where nuclear structure effects become clearly visible. A particularly precise measurement has been made of the difference between the $1S-2S$ transition of deuterium and hydrogen [34]:

$$\nu_{exp} = 670\,994\,334.64(15)\,\mathrm{kHz}. \tag{24}$$

Currently this measurement serves as the most precise determination of the deuteron structure radius, since all the QED effects are known with sufficient accuracy.

The contributions to the H-D isotope shift are:

- difference in the reduced mass
- higher order recoil corrections
- finite nuclear size
- deuteron polarizability

The most significant contribution is due to the different masses of the nuclei. The leading effects [12] are known to be

$$E = \mu\,(\mathcal{E}(n,j)-1) - \frac{\mu^2}{2\,(M+m)}\,(\mathcal{E}(n,j)-1)^2\,, \tag{25}$$

and amount to 671 004 073.56 kHz. Here $\mathcal{E}(n,j)$ denotes the dimensionless Dirac energy and m, M, μ are the electron, nucleus and reduced mass respectively. All other corrections are a few orders of magnitude smaller, but are nevertheless important for comparison with experiment. The recoil corrections come from the mass dependence of the Lamb shift and we have already described them in the previous section. They contribute -4 536.16 kHz to the isotope shift. There is an additional recoil correction due to the fact that the deuteron has spin 1 rather than 1/2 [35]. It is given by $\Delta E = -\mu^3\alpha^4/(2\,M^2\,n^3)$, and contributes 11.37 kHz for the H-D shift. The presence of this correction depends on the actual definition of the charge radius for a spin 1 particle.

The nuclear size correction presents the most difficult problem in the calculation of the H-D shift. If we use a standard definition of the charge radius,

$$\frac{\langle r^2\rangle}{6} = \frac{\partial G_E(q^2)}{\partial(q^2)}\bigg|_{q^2=0}, \tag{26}$$

we encounter a conceptual problem with the nuclear self–energy correction. Part of it is included already in the charge radius. Moreover, radiative corrections to G_E are infrared divergent. Therefore, the definition in Eq. (26) is no longer proper at the current level of precision. One can use the enhanced definition proposed in [17], which is based on the forward scattering amplitude. With this definition the correction to the energy for S–states takes the form

$$\begin{aligned}\Delta E = &\frac{2}{3\,n^3}\,(Z\,\alpha)^4\,\mu^3\langle r^2\rangle \\ &+\frac{4\,(Z^2\,\alpha)\,(Z\,\alpha)^4}{3\,\pi\,n^3}\,\frac{\mu^3}{M^2}\left[\ln\left(\frac{M}{\mu\,(Z\,\alpha)^2}\right) - \ln k_0(n)\right],\end{aligned} \tag{27}$$

where $\ln k_0(n)$ is the Bethe logarithm and Z denotes the nucleus charge. The first term is due to the finite nuclear size, while the second term comes from the proton self–energy and contributes 2.98 kHz to the H–D isotope shift. There are several further structure corrections, for example, relativistic and self energy corrections combined with the finite size effect. The most interesting one is due to the polarizability of the deuteron. The binding energy of the deuteron of 2.2 MeV is comparable to the electron rest energy, giving rise to a observable polarizability correction. It was first estimated using a simple nucleon-nucleon potential [36]. Later it was found that this correction does not significantly depend on the interaction potential [37]. The most recent analysis of Friar *et al.* [38] gives 18.58(7) kHz for the 1S–2S frequency in deuterium.

Summing up all the known theoretical contributions and *excluding* nuclear size effects, the theoretical value for the isotope shift is [34]

$$\Delta\nu_{th} = 670\,999\,568.6\,(1.5)(1.5)\,\text{kHz}, \tag{28}$$

where the first uncertainty comes from the electron–proton mass ratio and the second is the theoretical uncertainty. The difference between the theoretical and the experimental value is caused by the nuclear finite–size effect, which is given by the first term of Eq. (27). From this one derives the deuteron structure radius [34]

$$r_{str} = 1.97535(85)\ \text{fm}\,. \tag{29}$$

This determination exceeds the accuracy of previous measurements [39] based on elastic electron scattering by an order of magnitude. Furthermore, this result is

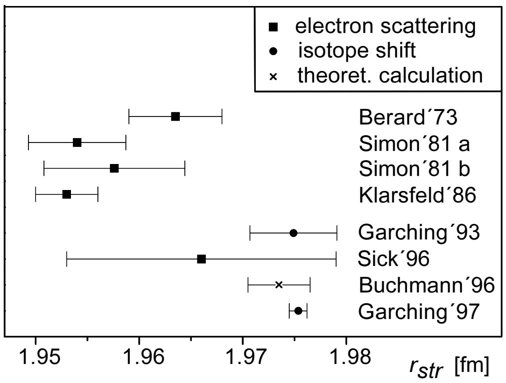

Figure 4. *Values for the deuteron structure radius.*

in disagreement with these measurements, as can be seen from Fig. (4), (taken from [34]). A recent reanalysis of the scattering data by Sick [40], in which the second order Born approximation has been included, indicates a possible agreement with (29). Moreover, the theoretical prediction of the deuteron structure radius by Buchmann et al. [41] also agrees with (29). Let me emphasize that this determination of r_{str} requires a careful analysis of the nuclear self–energy and the associated definition of the nuclear charge radius.

4. Ground state hyperfine structure in muonium and hydrogen

The hyperfine structure in the ground state comes from the interaction of the electron spin with the spin of the nucleus. It has been found by Fermi that the splitting of the ground state in leading order is given by the contact interaction:

$$V_F = \frac{e^2\, g_p}{3\, m_e\, m_p}\, \delta^3(r)\, \mathbf{s}_e \cdot \mathbf{s}_p\,, \tag{30}$$

To compare the the most precise measurement in hydrogen [4]

$$E_{exp} = 1420.405\,751\,766\,7(9)\,\mathrm{MHz}\,, \tag{31}$$

one considers various QED corrections. However, the proton structure correction as given by [12]:

$$\Delta E = \frac{64}{3}\, \frac{(Z\,\alpha)^5\, \mu^3}{\pi\, m_p} \int_0^\infty \frac{dp}{p^2} \Big(G_M(-p^2)\, G_E(-p^2) - G_M(0)\, G_E(0) \Big)\,, \tag{32}$$

require the experimental determination of G_E and G_M. These problems do not appear in muonium, which consists of the electron and the point-like muon. This purely leptonic atom is an ideal system for theorists, since all non QED effects are very small, and can be accounted for with sufficient accuracy. The current experimental value of muonium hfs is [5]

$$E_{exp} = 4\,463\,302.88(16)\,\mathrm{kHz}\,. \tag{33}$$

On the other hand, the theoretical prediction is primarily limited by the electron-muon mass ratio. A new improved measurement of this mass ratio and muonium hfs [42] is in preparation, which has stimulated the evaluation of higher order QED corrections. There were two major achievements in recent years. The first was the calculation of two–loop contribution of by Kinoshita [43] and by Eides *et al* [20] of order $\alpha^2 (Z\,\alpha)\, E_F$.

$$\Delta\nu = E_F\,(1 + a_\mu)\, \left(\frac{\alpha}{\pi}\right)^2 \left\{ a_e^{(4)} + [0.7717(4)]\, \pi\,(Z\,\alpha) - \frac{4}{3}\, \ln^2(Z\,\alpha)\,(Z\,\alpha)^2 \right\} \tag{34}$$

The second achievement was the precise calculation of the one–loop contribution in order $\alpha (Z\,\alpha)^2\, E_F$ in [44] and [45], and also to all orders by Blundell *et al.* in [46].

$$\begin{aligned} \Delta\nu = E_F\,(1 + a_\mu)\, \frac{\alpha}{\pi} \bigg\{ & \frac{1}{2} + \pi\,(Z\,\alpha) \Big(\ln(2) - \frac{5}{2} \Big) - \\ & \frac{8\,(Z\,\alpha)^2}{3}\, \ln(Z\,\alpha) \Big(\ln(Z\,\alpha) - \ln(4) + \frac{281}{480} \Big) + (Z\,\alpha)^2\, 15.922(15) \bigg\}, \end{aligned} \tag{35}$$

where the last term incorporates also all higher order terms for the self–energy, and leading terms for vacuum polarization. When one includes also the recoil corrections, then the theoretical predictions for the ground state muonium hyperfine splitting become [46]:

$$\Delta\nu = 4\,463\,302.89(1.33)(0.03)\,\mathrm{kHz}\,, \tag{36}$$

where the first error comes from the mass ratio and the second from α. This result contains a small contribution from the weak interaction of -0.065 kHz. A more precise value for the muon mass is needed for the improvement of this QED test. Alternatively, the *hfs* measurement may serve as the best muon mass determination.

5. Positronium energy levels

Positronium is the simplest atomic system that we can deal with. The small mass of it's constituents, the electron and positron, ensures that strong and weak interaction effects contribute at a negligible level. The energy spectrum may therefore be predicted with an accuracy limited only by the complexity in higher order QED calculations. The equal mass of the electron and the positron requires special theoretical treatment to incorporate relativistic and recoil effects on the same footing. Several approaches have been developed for the calculation of the positronium spectrum. The traditional one is based on the Bethe–Salpeter equation. This equation is a generalization of the Dirac equation for the case where the second particle is not static. However, the global wave function has an extra dependence on relative time, and therefore loses a probabilistic interpretation. It serves as a tool to calculate bound state energies. The cost one pays using this very general approach is the complicated implementation of the perturbation scheme. Nevertheless this method has been successfully used by Adkins *et. al* for the calculation of various corrections to the decay rate [11] and the hyperfine structure [47–49]. An alternative approach is based on the idea of introducing some effective interaction, which takes advantage of the cancellation mechanism between various corrections. Such an approach was first introduced by Caswell and Lepage [21]. It is called NRQED: nonrelativistic quantum electrodynamics. It contains many higher order couplings designed to match the low energy limit of the complete theory. This approach, after being applied to the calculation of the three–photon exchange contribution to *hfs* in positronium, has recently been used for the calculation of muonium *hfs* [45] and the one–photon annihilation channel contributions to the positronium *hfs* [50]. A similar approach has been introduced by me in [22,51], which is an extension of the Breit Hamiltonian to higher orders. One constructs an effective two–body Hamiltonian H_{EF} from the Hamiltonian of complete QED H_{QED} by comparing corresponding resolvents to the specified order n in α:

$$\left\langle \mathbf{p}', -\mathbf{p}' \left| \frac{1}{E - H_{QED}} \right| \mathbf{p}, -\mathbf{p} \right\rangle^{(n)} = \lim_{\lambda \to \infty} \left\langle \mathbf{p}' \left| \frac{1}{E - H_{EF}(\lambda, E)} \right| \mathbf{p} \right\rangle^{(n)} \tag{37}$$

In this comparison one assumes that momenta and energy are of the order $m\,\alpha$ and $m\,\alpha^2$ respectively. The additional parameter λ is introduced to ensure that matrix elements of H_{EF} are finite. The actual calculation of H_{EF} is split into two parts. The low energy part is given by time ordered perturbation theory diagrams, as shown in Fig. (5). The details of these calculations may be found in Ref. [22]. The

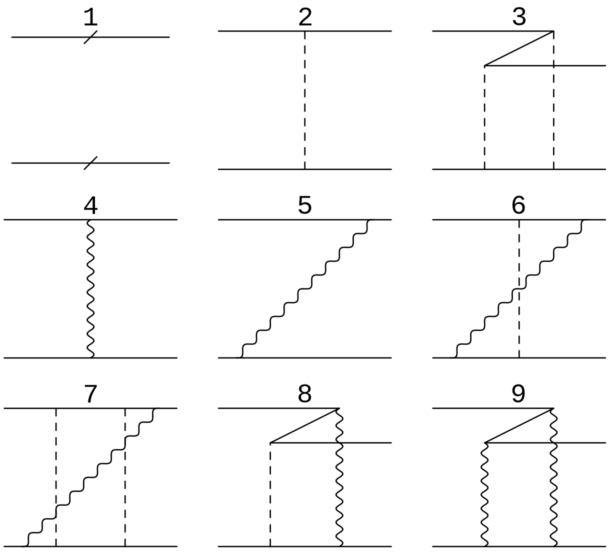

Figure 5. Time ordered diagrams contributing to positronium energy levels in order $m\,\alpha^6$

high energy part is given by the forward scattering amplitude involving exchange of three photons. When both parts are added, the contribution to energy does not depend on extra parameter λ. We have used this approach for the calculation of recoil effects in positronium energy levels. However, the result for the ground state hyperfine structure is in serious disagreement with the former calculation of Caswell and Lepage [21] (0.3767(17) versus 0.167(33)), and this problem remains to be resolved. Many other contributions have been calculated over the years.

The two– and three–photon annihilation channel has been calculated by Adkins in Ref's [47,48]. One photon annihilation has been recently obtained by Adkins *et al* [49] and independently by Hoang *et al* [50]. Together with S. Karshenboim, we have completed the theoretical predictions for positronium energy levels through order $m\,\alpha^6$ by the calculation of the radiative recoil correction [52]. The final results for the 1S and 2S states, using Ref. [22] are:

$$E^{(6)}(1S) = m\,\alpha^6\,[0.16107(63) - 0.3925(17)\,\mathbf{s}_-\mathbf{s}_+]\,, \tag{38}$$

$$E^{(6)}(2S) = \frac{m\,\alpha^6}{8}\,[0.34557(63) - 0.1048(17)\,\mathbf{s}_-\mathbf{s}_+]\,, \tag{39}$$

where s_- and s_+ are spin operators for electron and positron respectively. The theoretical predictions are compared to several precisely measured transitions in Table I (taken from [52]). Because of the discrepancy between the two calculations of three photon exchange, we cannot draw final conclusions. However, the Caswell, Lepage result is in agreement, and my result in $3.4\,\sigma$ disagreement with the measurement of *hfs*. Other transitions are not significantly affected by this discrepancy. For example the 2^3S_1 state might be shifted by 0.01 MHz, and in all cases theoretical predictions are now more precise than corresponding measurements.

6. Summary

Quantum electrodynamics gives a self–consistent description of atomic systems. It has been successfully developed for hydrogenic atoms, such us hydrogen, muonium and positronium. Its predictions in most cases are in agreement with the experimental results. However a few disagreements, specifically the He^+ Lamb shift and *hfs* and decay rates in positronium, remain to be resolved. The QED of many electron atoms has not yet been as well developed. Particularly interesting is the helium atom, where several precise spectroscopic measurements have recently been performed [61]. Theoretical predictions for helium energy levels are an order of magnitude less precise, because of difficulties in calculating multi photon exchange contributions. I believe the effective Hamiltonian approach, which was briefly presented here, is the most suitable method for the helium atom. In fact, a complete set of operators has already been derived for helium energy levels at order $m\,\alpha^6$. The problem now is the numerical one, of the calculation of the corresponding matrix elements. Recently particular atten-

Transition	Exp. Refs.	Experiment [MHz]	Theory [MHz]	Difference [MHz]
$2^3S_1 - 1^3S_1$	[53]	1233 607 218.9 (10.7)	1233 607 221.0 (1.0)	2.1(10.7)(1.0)
	[7]	1233 607 216.4 (3.2)		4.6(3.2)(1.0)
$1^3S_1 - 1^1S_0$	[54]	203 387.5 (1.6)	203 388.09 (0.80) ^{a}	0.6(1.6)(0.8) ^{a}
			203 392.02 (0.50) ^{b}	4.5(1.6)(0.5) ^{b}
	[6]	203 389.10 (0.74)		-1.01(0.74)(0.80) ^{a}
				2.92(0.74)(0.50) ^{b}
$2^3S_1 - 2^3P_0$	[55]	18504.1 (10.0)(1.7)	18498.42(0.13)	-5.7(10.0)(1.7)
	[56,57]	18499.65 (1.20)(4.00)		-1.2(1.2)(4.0)
$2^3S_1 - 2^3P_1$	[55]	13001.3 (3.9)(0.9)	13012.58(0.13)	11.3(3.9)(0.9)
	[56,57]	13012.42(0.67)(1.54)		0.2(0.7)(1.5)
$2^3S_1 - 2^3P_2$	[55]	8619.6(2.7)(0.9)	8626.87(0.13)	7.3(2.7)(0.9)
	[58]	8628.4(2.8)		-1.5(2.8)
	[56,57]	8624.38(0.54)(1.40)		2.5(0.5)(1.4)
$2^3S_1 - 2^1P_1$	[59]	11181(13)	11185.54(0.13)	5(13)
	[57]	11180(5)(4)		6(5)(4)
$2^3S_1 - 2^1S_0$	not measured yet		25424.69(0.06) ^{b}	

Table 1
Comparison of experiments to theoretical predictions. References to conflicting theoretical works are: ^{a} – [21]; ^{b} – [22]. Theoretical uncertainties are dominated by estimates for unknown higher order terms of order $m\alpha^7$. The differences for $2S - 2P$ are presented without theoretical uncertainties, since they are significantly smaller than experimental ones. The levels are labeled as $n^{2S+1}L_J$.

tion has been focussed on the fine structure of n^3P_J-levels. Several experimental groups are attempting to measure it with a precision allowing for the determination of the fine structure constant α with a precision better than that available from solid state experiments. This will require the calculation of $m\,\alpha^7$ corrections, which is a highly nontrivial task.

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