Theory of the Lamb shift in hydrogen and light hydrogen-like ions

Vladimir A. Yerokhin
Center for Advanced Studies, Peter the Great St. Petersburg Polytechnic University, Polytekhnicheskaya 29, St. Petersburg 195251, Russia

Krzysztof Pachucki
Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland

Vojtěch Patkóš
Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 121 16 Prague 2, Czech Republic

Theoretical calculations of the Lamb shift provide the basis required for the determination of the Rydberg constant from spectroscopic measurements in hydrogen. The recent high-precision determination of the proton charge radius drastically reduced the uncertainty in the hydrogen Lamb shift originating from the proton size. As a result, the dominant theoretical uncertainty now comes from the two- and three-loop QED effects, which calls for further advances in their calculations. We review the present status of theoretical calculations of the Lamb shift in hydrogen and light hydrogen-like ions with the nuclear charge number up to $Z = 5$. Theoretical errors due to various effects are critically examined and estimated.

I. INTRODUCTION

Hydrogen atom plays a special role in modern physics. As the simplest atomic system, hydrogen is often considered to be an ideal testing ground for exploring limits of the theory based on predictions of the bound-state quantum electrodynamics (QED). One of the important tests of theory is the comparison of the proton charge radius values obtained from the Lamb shift in electronic and muonic hydrogen. The $4.5\,\sigma$ discrepancy between these values, known as the proton radius puzzle [1, 2], attracted large attention of the scientific community. This discrepancy could indicate violation of the lepton universality and existence of interactions not accounted for in the Standard Model. Such a possibility is still open, although recent experiments on electronic hydrogen [3–6] hint at existence of unknown systematic effects in hydrogen measurements rather than at new physics.

Another important role of hydrogen is that comparison of theory and experiment for its transition energies is used [7] for determining the Rydberg constant, which is one of the most accurately known fundamental constants today. If one adopts the proton charge radius determined from the muonic hydrogen [2], the uncertainty of the Rydberg constant is defined by the currently available theory of the hydrogen Lamb shift.

Precise spectroscopy of light hydrogen-like ions may also provide determinations of the Rydberg constant in the foreseeable future. Such determinations will be independent on the proton radius and systematic effects in the hydrogen spectroscopy. Helium isotopes look most promising in this respect, because of high-precision results for nuclear radii expected soon from experiments on muonic helium [8]. We mention here the ongoing projects of measuring the $1S$–$2S$ transition energy in He^+ pursued in Garching [9] and in Amsterdam [10], which require improved theoretical predictions for the helium Lamb shift.

Motivated by the needs outlined above, in the present work we summarize the presently available theory for the Lamb shift of hydrogen and light hydrogen-like ions with the nuclear charge up to $Z = 5$. This summary is intended as an update of the CODATA review of the hydrogen theory [7]. In particular, we perform a reanalysis of results available for the higher-order two-loop QED corrections, which presently define the theoretical uncertainty of the Lamb shift. Results for the nuclear recoil effect are significantly improved by taking into account recent nonperturbative calculations [11, 12]. The nuclear finite size and nuclear polarizability effects are reformulated, according to recent theoretical developments [13, 14].

Relativistic units $m = \hbar = c = 1$ are used throughout this paper (where m is the electron mass). In these units the electron rest mass energy $mc^2 = 1$, so that all energy corrections appear to be dimensionless. In order to convert any energy correction in relativistic units to arbitrary units, it is sufficient to multiply it by $2\mathcal{R}/\alpha^2$, where $\mathcal{R} = hcR_\infty$ is the Rydberg energy and R_∞ is the Rydberg constant. While $m = 1$ in our units, we will write m explicitly when it enters dimensionless ratios, such as m/M and m_r/m.

II. BINDING ENERGY

We consider the binding energy E_{njl} of an electronic state with quantum numbers n, j, and l in a light hydrogen-like atom. If the atomic nucleus has a nonzero spin I, the energy level $|njl\rangle$ is splitted by the interaction with the nuclear magnetic moment according to values of the total angular momentum F, $|njlF\rangle$. In this case, we define the binding energy E_{njl} as a centroid averaged over all hyperfine-structure (hfs) components,

$$E_{njl} = \frac{\sum_F (2F+1)\, E_{njlF}}{\sum_F (2F+1)}\,. \tag{1}$$

The interaction with the dipole nuclear magnetic moment, responsible for the hyperfine structure, does not contribute to E_{njl} in the first order. There is, however, a second-order hfs effect that shifts (slightly) the centroid energy E_{njl}. It manifests itself as a nuclear-spin dependent recoil correction and is addressed in Sec. IV.

The goal of the present paper is to summarize the presently available theory for the binding energy E_{njl} of the $1S$, $2S$, and $2P_{1/2}$ states of light hydrogen-like atoms. The hyperfine splitting of energy levels will not be discussed. For the nS states it was investigated in detail in Ref. [15]; a review of the hfs of the higher-l states is available in Ref. [16].

The binding energy of a light hydrogen-like atom is usually represented as a sum of three contributions,

$$E_{njl} = E_D + E_M + E_L\,, \tag{2}$$

where E_D is the Dirac point-nucleus biding energy in the nonrecoil limit, E_M is the correction containing the dominant part of the nuclear recoil effect, and E_L is the Lamb shift. We note that the total recoil effect is thus distributed between E_M and E_L (E_M being the dominant part and smaller corrections being ascribed to the Lamb shift E_L). This distribution is not unique and done differently in the literature.

The Dirac point-nucleus nonrecoil binding energy E_D is given by

$$E_D = \sqrt{1-\frac{(Z\alpha)^2}{N^2}} - 1\,, \tag{3}$$

where

$$N = \sqrt{(n_r+\gamma)^2+(Z\alpha)^2}\,, \tag{4}$$

$\gamma = \sqrt{\kappa^2-(Z\alpha)^2}$, $n_r = n-|\kappa|$ is the radial quantum number, n is the principal quantum number, and $\kappa = (l-j)(2j+1)$ is the angular momentum-parity quantum number.

The leading recoil correction E_M is

$$E_M = \frac{m}{M}\frac{(Z\alpha)^2}{2\,N^2} - \left(\frac{m}{M}\right)^2\frac{(Z\alpha)^2}{2\,n^2}\,\frac{m_r}{m}\,, \tag{5}$$

where M is the nuclear mass and $m_r = mM/(m+M)$ is the reduced mass. All further recoil corrections are ascribed to the Lamb shift E_L. The first part of E_M comprises the complete m/M recoil effect to orders $(Z\alpha)^2$ and $(Z\alpha)^4$ and, in addition, corrections of order $(Z\alpha)^6$ and higher that can be obtained from the Breit Hamiltonian. The second part of E_M is the nonrelativistic recoil correction of second and higher orders in m/M. In the nonrelativistic limit, the sum E_D+E_M reduces to the Schrödinger energy eigenvalue,

$$E_D + E_M = \frac{m_r}{m}\frac{(Z\alpha)^2}{2\,n^2} + \dots\,, \tag{6}$$

where $\dots$ represents contributions of order $(Z\alpha)^4$ and higher.

Our choice of E_M (and, therefore, our definition of the Lamb shift E_L) follows Ref. [17] and differs slightly from the popular definition [18] based on the Barker-Glover formula [19] and, as a consequence, from the definition of the CODATA review [7] (*cf.* Eqs. (25) and (26) therein). The reason for this difference was the need for a simple and concise definition valid for an arbitrary nucleus, whereas the Barker-Glover formula is valid only for the spin-1/2 nucleus. Both definitions are equivalent through orders $(m/M)(Z\alpha)^{2+n}$ and $(m/M)^{2+n}(Z\alpha)^2$, with $n \geq 0$. The difference is that our present definition of Eq. (5) does not contain any contribution of order $(m/M)^2(Z\alpha)^4$ (which depends on the nuclear spin) or any spurious higher-order terms. The correction of order $(m/M)^2(Z\alpha)^4$ is included into the Lamb shift; it is given by the first line of Eq. (41).

Another difference in definitions in the literature is associated with the off-diagonal hfs correction, which is small but relevant on the level of the experimental interest for the $l > 0$ states [20]. In the old Lamb-shift measurements (in particular, Ref. [21]), this correction was subtracted from the experimental result. Reviews [7, 17] do not discuss it, thus excluding it from the definition of the Lamb shift. The review [16] includes this correction [see Eq. (30) therein] but ascribes it to the hyperfine splitting. A part of the off-diagonal hfs correction shifts the centroid energy E_{njl} and thus needs to be included into the definition of the Lamb shift. The corresponding contribution is given by Eq. (42).

We now turn to examining various effects that contribute to the Lamb shift E_L.

III. QED EFFECTS

A. One-loop QED effects

The one-loop QED effects for the point nuclear charge are represented as

$$E_{\rm QED1} = \frac{\alpha}{\pi}\frac{(Z\alpha)^4}{n^3}\left(\frac{m_r}{m}\right)^3 \times \Big[F_{\rm SE}(Z\alpha) + F_{\rm VP}(Z\alpha)\Big]\,, \tag{7}$$

where the functions $F_{\rm SE}(Z\alpha)$ and $F_{\rm VP}(Z\alpha)$ correspond to the one-loop self-energy and vacuum-polarization, respectively.

The $Z\alpha$ expansion of the electron self-energy is given by

$$F_{\rm SE}(Z\alpha) = L\,A_{41} + A_{40} + (Z\alpha)\,A_{50} + (Z\alpha)^2\left[L^2\,A_{62} + L\,A_{61} + G_{\rm SE,pnt}(Z\alpha)\right], \tag{8}$$

where $L = \ln\left[(m/m_r)(Z\alpha)^{-2}\right]$ and $G_{\rm SE}(Z\alpha) = A_{60} + \dots$ is the remainder that contains all higher-order expansion terms in $Z\alpha$. The coefficients of the $Z\alpha$ expansion in Eq. (8) are well known. They are discussed, e.g., in review [22] and summarized in Table I. Numerical results for the remainder function are obtained by Jentschura and Mohr [23, 24] and listed in Table II. Results for $Z = 0$ correspond to the coefficient A_{60}; they were taken from Ref. [25].

The $Z\alpha$ expansion of the vacuum-polarization correction is given by

$$F_{\rm VP}(Z\alpha) = -\frac{4}{15}\,\delta_{\ell 0} + \frac{5}{48}\pi(Z\alpha)\,\delta_{\ell 0} + (Z\alpha)^2 \times \left[-\frac{2}{15}L\,\delta_{\ell 0} + G_{\rm Ueh}(Z\alpha) + G_{\rm WK}(Z\alpha)\right], \tag{9}$$

where $G_{\rm Ueh}(Z\alpha)$ and $G_{\rm WK}(Z\alpha)$ are the higher-order remainder functions induced by the Uehling and Wichmann-Kroll parts of the vacuum polarization, respectively. Numerical results for the remainder functions are listed in Table II. The Wichmann-Kroll part of the vacuum polarization was calculated with help of the approximate potential based on the analytical expansions of Whittaker functions from Ref. [26]. The uncertainty due to approximations in the potential is negligible at the level of current interest. In the limit $Z \to 0$, results for the higher-order remainders are (see review [22] for details)

$$G_{\rm Ueh}(Z=0,1S) = \frac{4}{15}\Big(\ln 2 - \frac{1289}{420}\Big)\,, \tag{10}$$

$$G_{\rm Ueh}(Z=0,2S) = -\frac{743}{900}\,, \tag{11}$$

$$G_{\rm Ueh}(Z=0,2P_{1/2}) = -\frac{9}{140}\,, \tag{12}$$

$$G_{\rm WK}(Z=0) = \Big(\frac{19}{45} - \frac{\pi^2}{27}\Big)\,\delta_{\ell 0}\,. \tag{13}$$

The vacuum-polarization induced by the $\mu^+\mu^-$ virtual pairs is given by [27, 28]

$$E_{\mu\rm VP} = \left(\frac{m}{m_\mu}\right)^2 \frac{\alpha}{\pi}\,\frac{(Z\alpha)^4}{n^3}\,\left(\frac{m_r}{m}\right)^3\,\left(-\frac{4}{15}\right)\,\delta_{\ell 0}\,, \tag{14}$$

where m_μ is the muon mass.

The hadronic vacuum-polarization correction is of the same order as the muonic vacuum polarization and is given by [29]

$$E_{\rm hadVP} = 0.671\,(15)\,E_{\mu\rm VP}\,. \tag{15}$$

B. Two-loop QED effects

The two-loop QED correction is expressed as

$$E_{\rm QED2} = \Big(\frac{\alpha}{\pi}\Big)^2\,\frac{(Z\alpha)^4}{n^3}\,\left(\frac{m_r}{m}\right)^3\,F_{\rm QED2}(Z\alpha)\,, \tag{16}$$

where the function $F_{\rm QED2}$ is given by

$$F_{\rm QED2}(Z\alpha) = B_{40} + (Z\alpha)\,B_{50} + (Z\alpha)^2\big[B_{63}\,L^3 + B_{62}\,L^2 + B_{61}\,L + G_{\rm QED2}(Z\alpha)\big]\,, \tag{17}$$

and $G_{\rm QED2}(Z\alpha) = B_{60} + \ldots$ is the remainder that contains all higher-order expansion terms in $Z\alpha$.

The two-loop QED correction is conveniently divided into three parts: the two-loop self-energy (SESE), the two-loop vacuum-polarization (VPVP), and the mixed self-energy and vacuum-polarization (SEVP),

$$F_{\rm QED2} = F_{\rm SESE} + F_{\rm SEVP} + F_{\rm VPVP}\,. \tag{18}$$

Coefficients of the $Z\alpha$ expansion of the individual two-loop corrections for the states under consideration are summarized in Table III, for details see recent studies [25, 30–34] and references to earlier works therein. We note that the analytical result for the B_{61} coefficient derived in Ref. [30] was incomplete; one missing piece was added later in Ref. [25] and another, in Ref. [34]. The listed value of B_{61} for the S states differs from that given in Refs. [7, 17] by $-43/36 + 133\pi^2/864 = -0.134567\ldots$, which is the light-by-light contribution from Ref. [34]. Numerical values for the delta-function correction to the Bethe logarithm $\mathcal{N}(nS)$ and $\mathcal{N}(nP)$ that enter B_{61} can be found in Refs. [25, 35].

The two-loop higher-order remainder $G_{\rm QED2}$ is only partly known up to now. Its $Z\alpha$ expansion has the form

$$G_{\rm QED2}(Z\alpha) = B_{60} + (Z\alpha)\big[B_{72}\,L^2 + B_{71}\,L + \ldots\big]\,. \tag{19}$$

The dominant part of the coefficient B_{60} comes from the two-loop self-energy. It was calculated for the $1S$ and $2S$ states by Pachucki and Jentschura [31], with the result

$$B_{60}(1S,\rm SESE) = -61.6\,(9.2)\,, \tag{20}$$

$$B_{60}(2S,\rm SESE) = -53.2\,(8.0)\,, \tag{21}$$

where the uncertainty comes from omitted contributions. The complete n dependence of $B_{60}(nS)$ was calculated in Refs. [25, 32]. For the nP states, the coefficient B_{60} was calculated in Ref. [36]. The results for the SESE and SEVP corrections and the $2P_{1/2}$ state are

$$B_{60}(2P_{1/2},\rm SESE) = -1.5\,(3)\,, \tag{22}$$

$$B_{60}(2P_{1/2},\rm SEVP) = -0.016\,571\ldots\,. \tag{23}$$

We use opportunity to correct a mistake in Ref. [36] for the VPVP correction (given by Eqs. (A3) and (A6) of that work). The corrected results are

$$B_{60}(nP_{1/2},\rm VPVP) = -\frac{713}{2025}\left(1 - \frac{1}{n^2}\right)\,, \tag{24}$$

$$B_{60}(nP_{3/2},\rm VPVP) = -\frac{401}{4050}\left(1 - \frac{1}{n^2}\right)\,. \tag{25}$$

The logarithmic coefficients B_{72} and B_{71} in Eq. (19) were recently investigated in Ref. [37]. The leading logarithmic coefficient B_{72} was derived as

$$B_{72}(\rm SESE) = \Big(-\frac{139}{48} + \frac{4}{3}\ln 2\Big)\,\pi\,\delta_{\ell 0}\,, \tag{26}$$

$$B_{72}(\rm SEVP) = -\frac{5}{72}\,\pi\,\delta_{\ell 0}\,, \tag{27}$$

$$B_{72}(\rm VPVP) = 0\,. \tag{28}$$

The next coefficient B_{71} was obtained for the nP states, with the result

$$B_{71}(\rm SESE, nP) = \Big(\frac{139}{144} - \frac{4}{9}\ln 2\Big)\,\pi\,\frac{n^2-1}{n^2}\,, \tag{29}$$

$$B_{71}(\rm SEVP, nP) = \frac{5}{216}\,\pi\,\frac{n^2-1}{n^2}\,, \tag{30}$$

$$B_{71}(\rm VPVP, nP) = 0\,. \tag{31}$$

Ref. [37] also reported the n dependence of $B_{71}(nS)$.

TABLE I: Coefficients of the $Z\alpha$ expansion of the one-loop electron self-energy in Eq. (8).

Term		$1S$	$2S$	$2P_{1/2}$
A_{41}	$\frac{4}{3}\,\delta_{\ell 0}$	$\frac{4}{3}$	$\frac{4}{3}$	0
A_{40}	$-\frac{4}{3}\,\ln k_0(n,l) + \frac{10}{9}\,\delta_{\ell 0} - \frac{m/m_r}{2\kappa(2l+1)}(1-\delta_{\ell 0})$	$-2.867\,726\,964$	$-2.637\,915\,413$	$-0.126\,644\,388\,(m/m_r)$
A_{50}	$\left(\frac{139}{32} - 2\ln 2\right)\,\pi\,\delta_{\ell 0}$	$9.291\,120\,908$	$9.291\,120\,908$	0
A_{62}	$-\delta_{\ell 0}$	-1	-1	0
A_{61}	$4\Big(\frac{4}{3}\ln 2 + \ln\frac{2}{n} + \psi(n+1) - \psi(1) - \frac{601}{720} - \frac{77}{180n^2}\Big)\delta_{\ell 0}$ $+\left[\frac{n^2-1}{n^2}\left(\frac{2}{15} + \frac{1}{3}\delta_{j,1/2}\right) + 8\frac{3-l(l+1)/n^2}{3(2l+3)l(l+1)(4l^2-1)}\right](1-\delta_{\ell 0})$	$5.419\,373\,685$	$5.930\,118\,296$	$0.572\,222\,222$

TABLE II: Results for the higher-order remainder functions $G_{\rm SE}$, $G_{\rm Ueh}$, and $G_{\rm WK}$ in Eqs. (8) and (9).

Z	$1S$	$2S$	$2P_{1/2}$
Self-energy:			
0	$-30.924\,149\,46\,(1)$	$-31.840\,465\,09\,(1)$	$-0.998\,904\,40$
1	$-30.290\,24\,(2)$	$-31.185\,15\,(9)$	$-0.973\,45\,(19)$
2	$-29.770\,967\,(5)$	$-30.644\,66\,(5)$	$-0.949\,40\,(5)$
3	$-29.299\,170\,(2)$	$-30.151\,93\,(2)$	$-0.926\,37\,(2)$
4	$-28.859\,222\,(1)$	$-29.691\,27\,(1)$	$-0.904\,12\,(1)$
5	$-28.443\,372\,(1)$	$-29.255\,033\,(8)$	$-0.882\,478\,(8)$
Vacuum-polarization, Uehling:			
0	$-0.633\,573$	$-0.825\,556$	$-0.064\,286$
1	$-0.618\,724$	$-0.808\,872$	$-0.064\,006$
2	$-0.607\,668$	$-0.796\,118$	$-0.063\,768$
3	$-0.598\,207$	$-0.785\,075$	$-0.063\,567$
4	$-0.589\,838$	$-0.775\,230$	$-0.063\,399$
5	$-0.582\,309$	$-0.766\,322$	$-0.063\,262$
Vacuum-polarization, Wichmann-Kroll:			
0	$0.056\,681$	$0.056\,681$	0
1	$0.055\,721$	$0.055\,721$	$0.000\,002$
2	$0.054\,823$	$0.054\,824$	$0.000\,006$
3	$0.053\,978$	$0.053\,983$	$0.000\,012$
4	$0.053\,178$	$0.053\,188$	$0.000\,020$
5	$0.052\,418$	$0.052\,437$	$0.000\,030$

Calculations of the SESE part of the higher-order remainder, $G_{\rm SESE}$, were carried out to all orders in $Z\alpha$ for hydrogenlike ions with $Z \geq 10$ [38, 39]. The latest results were obtained in Ref. [40] for $Z < 30$ and in Ref. [41] for $Z \geq 30$. The extrapolation of the all-order $1S$ results down to $Z = 1$ reported in Ref. [40] showed only a marginal agreement with the analytical value (20). A possible reason for this could be a large contribution from the unknown logarithmic coefficient B_{71}.

In the present work, we merge together the numerical and analytical results, in order to obtain the presumably best values for the higher-order remainder. Specifically, for the $1S$ state, we fit the numerical all-order data for $Z \geq 15$ from Refs. [40, 41] to the form

$$G_{\rm SESE}(1S) = B_{60} + B_{72}(Z\alpha)\ln^2(Z\alpha)^{-2} + b_{71}(Z\alpha)\ln(Z\alpha)^{-2} + (Z\alpha)\,{\rm pol}(Z\alpha)\,, \tag{32}$$

where B_{60} and B_{72} are given by Eqs. (20) and (26), and ${\rm pol}(Z\alpha)$ denotes a polynomial in $Z\alpha$. b_{71} and the coefficients of the polynomial are fitting parameters. The uncertainty was obtained by varying (*i*) B_{60} within its error bars (20), (*ii*) numerical data within their error bars, and (*iii*) the length of the polynomial and the number of data points included. The higher-order remainder for the $2S$ state was obtained by adding to $G_{\rm SESE}(1S)$ the difference $G_{\rm SESE}(2S) - G_{\rm SESE}(1S)$, as fitted in Ref. [41]. For the $2P_{1/2}$ state, we merged together the analytical results (22) and (29) and numerical data from Ref. [41]. The uncertainty was obtained by quadratically adding the error of the B_{60} coefficient and one half of the leading logarithmic B_{71} contribution. The obtained results for the higher-order SESE remainder are summarized in Table IV.

Calculations of the SEVP and VPVP corrections were performed in Ref. [42] to all orders in $Z\alpha$. Results for the higher-order remainder $G_{\rm SEVP}$ listed in Table IV were obtained from Tables I and IV of Ref. [42], after subtracting contributions of the leading $Z\alpha$-expansion coefficients and keeping in mind that the light-by-light (LBL) contribution was not included in numerical calculations and thus should not be subtracted. The

uncertainty of the SEVP contribution comes from the missing LBL contribution. It was estimated for the S states as one half of the LBL B_{61} contribution, calculated in Ref. [34]. For the P states, we assume the uncertainty to be negligible.

The results for the higher-order remainder $G_{\rm VPVP}$ listed in Table IV were obtained from Tables II and III of Ref. [42], after subtracting contributions of the leading $Z\alpha$-expansion coefficients summarized in Table III. The uncertainty due to omitted higher-order Källén-Sabry contributions is assumed to be negligible at the level of present interest.

For the $1S$ state of hydrogen, our result for the two-loop higher-order remainder is $G_{\rm QED2} = -92(13)$, which is slightly lower than the value adopted by CODATA 2016 of $-81(20)$ [7].

C. Higher-order QED effects

The $Z\alpha$ expansion of the three-loop QED correction is given by

$$E_{\rm QED3} = \left(\frac{\alpha}{\pi}\right)^3 \frac{(Z\alpha)^4}{n^3} \left(\frac{m_r}{m}\right)^3 \Big[C_{40} + (Z\alpha)\,C_{50} + (Z\alpha)^2 \Big(C_{62}\,L^2 + C_{61}\,L + \dots\Big)\Big]\,, \tag{33}$$

The leading-order contribution C_{40} was obtained in Refs. [43, 44] and is given by

$$\begin{aligned} C_{40} \;=\; & \Big[-\frac{568\,\mathrm{a}_4}{9} + \frac{85\,\zeta(5)}{24} \\ & - \frac{121\,\pi^2\,\zeta(3)}{72} - \frac{84\,071\,\zeta(3)}{2304} - \frac{71\,\ln^4 2}{27} \\ & - \frac{239\,\pi^2\,\ln^2 2}{135} + \frac{4787\,\pi^2\,\ln 2}{108} + \frac{1591\,\pi^4}{3240} \\ & - \frac{252\,251\,\pi^2}{9720} + \frac{679\,441}{93\,312}\Big]\delta_{\ell 0} \\ & + \Big[-\frac{100\,\mathrm{a}_4}{3} + \frac{215\,\zeta(5)}{24} \\ & - \frac{83\,\pi^2\,\zeta(3)}{72} - \frac{139\,\zeta(3)}{18} - \frac{25\,\ln^4 2}{18} \\ & + \frac{25\,\pi^2\,\ln^2 2}{18} + \frac{298\,\pi^2\,\ln 2}{9} + \frac{239\,\pi^4}{2160} \\ & - \frac{17\,101\,\pi^2}{810} - \frac{28\,259}{5184}\Big]\frac{m/m_r}{\kappa(2\ell+1)}(1-\delta_{\ell 0})\,, \end{aligned} \tag{34}$$

where $a_4 = \sum_{n=1}^{\infty} 1/(2^n\,n^4) = 0.517\,479\,061\ldots$. For the next-order contribution C_{50}, there are only partial results up to now [45, 46]. Following Ref. [7], we do not include partial results and estimate the uncertainty due to absence of this term as $C_{50} = \pm 30\,\delta_{\ell 0}$. The leading logarithmic contribution C_{62} was derived in Ref. [37] as

$$C_{62} = -\frac{2}{3}\,B_{40}\,, \tag{35}$$

where B_{40} is the leading-order two-loop coefficient summarized in Table III. Ref. [37] presented results also for the single-logarithmic contribution C_{61} for the nP states and the difference $C_{61}(nS) - C_{61}(1S)$.

IV. NUCLEAR RECOIL

The dominant part of the nuclear recoil effect is accounted for by E_M in Eq. (5) and by the reduced-mass prefactors in previous formulas. Beyond that, there are a number of further recoil corrections. The first one is the nuclear recoil correction of order $(Z\alpha)^{\geq 5}$ and of first order in m/M,

$$E_{\rm REC} = \frac{m}{M}\,\frac{(Z\alpha)^5}{\pi\,n^3}\left[\left(\frac{m_r}{m}\right)^3 \ln(Z\alpha)^{-2}\,D_{51} + \left(\frac{m_r}{m}\right)^3 D_{50} + (Z\alpha)\,D_{60} + (Z\alpha)^2\,G_{\rm REC}(Z\alpha)\right], \tag{36}$$

where $G_{\rm REC}(Z\alpha)$ is the higher-order remainder containing all higher orders in $Z\alpha$. Coefficients of the $Z\alpha$ expansion in Eq. (36) are reviewed in Ref. [22] and summarized in Table V. The higher-order remainder $G_{\rm REC}$ has an expansion of the form

$$G_{\rm REC}(Z\alpha) = D_{72}\,\ln^2(Z\alpha)^{-2} + D_{71}\,\ln^2(Z\alpha) + D_{70} + \dots\,, \tag{37}$$

where $D_{72} = -11/60\,\delta_{\ell 0}$ [47, 48] and the next two coefficients were obtained by fitting numerical results in Refs. [11, 12]

$$D_{71}(1S) = 2.919\,(10)\,, \quad D_{70}(1S) = -1.32\,(10)\,, \tag{38}$$

$$D_{71}(2S) = 3.335\,(10)\,, \quad D_{70}(2S) = -0.26\,(6)\,, \tag{39}$$

$$D_{71}(2P_{1/2}) = 0.149\,(5)\,, \quad D_{70}(2P_{1/2}) = -0.035\,(15)\,. \tag{40}$$

Numerical, all-order in $Z\alpha$ results for the higher-order remainder $G_{\rm REC}$ are obtained in Refs. [11, 12] and summarized in Table VI. In the present review we do not include results for the finite nuclear size correction to $E_{\rm REC}$ obtained in Refs. [11, 12], since this effect is partly included in calculations of nuclear polarizability summarized in the next section.

The relativistic recoil corrections of second order in the mass ratio is [18, 49, 50],

$$E_{\rm REC,2} = \left(\frac{m}{M}\right)^2 \frac{(Z\alpha)^4}{n^3}\left[\frac{3}{4n} - \frac{1}{2l+1} + \frac{1}{2}\,\delta_{\ell 0}\,\delta_{I,1/2} - (Z\alpha)\,\frac{2}{\pi}\left(1 + \frac{m}{M}\ln\frac{m}{M}\right)\delta_{\ell 0}\right]. \tag{41}$$

The first part of this correction $\propto (Z\alpha)^4$ depends on the nuclear spin I, which is the consequence of the choice of the definition of the point-like particle with a spin I. For $I > 1$ such a definition is not commonly established, so we ascribe an uncertainty of $\pm\frac{1}{2}\,\delta_{\ell 0}$ relative to the square brackets in the above formula. This part agrees with the $(Z\alpha)^4(m/M)^2$ term

TABLE III: Coefficients of the $Z\alpha$ expansion of the two-loop QED effects in Eq. (17). $\zeta(n)$ denotes the Riemann zeta function, $\psi(n)$ is the digamma function, γ_E is Euler's constant, $\mathcal{N}(nL)$ is a delta-function correction to the Bethe logarithm, defined by Eq. (4.21a) of Ref. [25].

Term		$1S$	$2S$	$2P_{1/2}$
SESE				
B_{40}	$\Big[-\frac{163}{72}-\frac{85}{216}\,\pi^2+\frac{3}{2}\,\pi^2\ln 2-\frac{9}{4}\,\zeta(3)\Big]\,\delta_{\ell 0}$ $-\Big[-\frac{31}{16}+\frac{5}{12}\,\pi^2-\frac{1}{2}\pi^2\,\ln 2+\frac{3}{4}\,\zeta(3)\Big]\frac{m/m_r}{\kappa\,(2\,l+1)}\,(1-\delta_{l0})$	1.409 244	1.409 244	$0.114\,722\,(m/m_r)$
B_{50}	unknown	$-24.265\,06\,(13)$	$-24.265\,06\,(13)$	0
B_{63}	$-\frac{8}{27}\,\delta_{\ell 0}$	$-8/27$	$-8/27$	0
B_{62}	$\frac{16}{9}\left(\frac{13}{12}-\ln 2+\frac{1}{4n^2}-\frac{1}{n}-\ln n+\psi(n)+\gamma_E\right)\delta_{\ell 0}$ $+\frac{4}{27}\,\frac{n^2-1}{n^2}\,\delta_{\ell 1}$	$-0.639\,669$	0.461 403	1/9
B_{61}	$\frac{4}{3}\,\mathcal{N}(nL)+\Big[\frac{15473}{2592}+\frac{1039}{432}\,\pi^2-\frac{152}{27}\,\ln 2-\frac{2}{3}\,\pi^2\,\ln 2+\frac{40}{9}\,\ln^2 2+\zeta(3)$ $+\Big(\frac{80}{27}-\frac{32}{9}\,\ln 2\Big)\Big(\frac{3}{4}+\frac{1}{4n^2}-\frac{1}{n}-\ln n+\psi(n)+\gamma_E\Big)\Big]\,\delta_{\ell 0}$ $+\frac{n^2-1}{n^2}\Big(\frac{11}{81}+\frac{1}{3}\delta_{j,1/2}-\frac{8}{27}\,\ln 2\Big)\,\delta_{\ell 1}$	48.388 913	40.932 915	0.202 220
SEVP				
B_{40}	$\Big(-\frac{7}{81}+\frac{5\pi^2}{216}\Big)\,\delta_{\ell 0}$ $+\Big(\frac{119}{36}-\frac{\pi^2}{3}\Big)\,\frac{j(j+1)-l(l+1)-3/4}{l(l+1)(2l+1)}\,\frac{m}{m_r}\,(1-\delta_{\ell 0})$	0.142 043	0.142 043	$-0.005\,229\,(m/m_r)$
B_{50}	unknown	1.305 370	1.305 370	0
B_{63}	0	0	0	0
B_{62}	$\frac{8}{45}\,\delta_{\ell 0}$	8/45	8/45	0
B_{61}	$\Big[-\frac{259}{1080}+\frac{41\,\pi^2}{432}+\frac{16}{15}\ln 2-\frac{32}{45}\,\Big(\frac{3}{4}+\frac{1}{4\,n^2}-\frac{1}{n}-\ln n+\psi(n)+\gamma_E\Big)\Big]\,\delta_{\ell 0}$ $-\frac{2}{45}\,\delta_{\ell 1}$	1.436 241	0.995 812	$-0.044\,444$
VPVP				
B_{40}	$-\frac{82}{81}\,\delta_{\ell 0}$	$-82/81$	$-82/81$	0
B_{50}	$\Big(\frac{7421-2625\pi}{6615}+\frac{52}{63}\ln 2\Big)\,\pi\,\delta_{\ell 0}$	1.405241	1.405241	0
B_{63}	0	0	0	0
B_{62}	0	0	0	0
B_{61}	$-\frac{1097}{2025}\,\delta_{\ell 0}$	$-0.541\,728$	$-0.541\,728$	0

contained in Eq. (25) of the CODATA review [7]. The second part of this correction $\propto (Z\alpha)^5$ is the Erickson formula (see the last line of Eq. (27) in Ref. [7]) expanded in m/M. This formula is derived for the spin-$1/2$ nucleus; its dependence on nuclear spin is not known. However, we assume the corresponding uncertainty to be negligible.

An additional recoil contribution arises for the P (and higher-l) states because of mixing of the fine-structure sublevels by the hyperfine-structure (hfs) interaction. This contribution is also known as the off-diagonal hfs shift. It depends on the nuclear spin I and the nuclear magnetic moment μ and is given, for the nP states [16, 20], by

$$E_{\rm REC,hfs}(nP) = \left(\frac{m}{m_p}\right)^2 \frac{\alpha^2(Z\alpha)^2}{n^3} \times \left(\frac{\mu}{\mu_N}\right)^2 \frac{2I(I+1)}{81}\,(-1)^{j+1/2}\,\delta_{\ell 1}\,, \tag{42}$$

where $\mu_N = |e|/(2\,m_p)$ is the nuclear magneton and m_p is the proton mass. This correction shifts the $2P_{1/2}$ centroid energy by -1.88 kHz for hydrogen, by -0.47 kHz for deuterium, and by -4.36 kHz for ^{3}He. We note that this correction was not included in the definition of the energy levels in the CODATA review [7] and needed to be accounted for together with the hyperfine structure. Corrections to Eq. (42) are assumed to be suppressed by α/π, which is included into uncertainty.

Furthermore, there is the radiative recoil correction [47, 51–53]

$$E_{\rm RREC} = \frac{m}{M}\left(\frac{m_r}{m}\right)^3 \frac{\alpha(Z\alpha)^5}{\pi^2 n^3}\,\delta_{\ell 0}\Big[6\,\zeta(3) - 2\pi^2\ln 2 + \frac{35\pi^2}{36} - \frac{448}{27} + \frac{2}{3}\,\pi(Z\alpha)\ln^2(Z\alpha)^{-2}\Big]\,. \tag{43}$$

Following Ref. [54], we ascribe to this correction an uncertainty of $10(Z\alpha)\ln(Z\alpha)^{-2}$ relative to the square brackets in the above equation.

TABLE IV: Results for the two-loop higher-order remainder $G_{\rm QED2}$ in Eq. (17).

Z	$1S$	$2S$	$2P_{1/2}$
SESE			
0	$-61.6\,(9.2)$	$-53.2\,(8.0)$	$-1.5\,(3)$
1	$-75.9\,(12.6)$	$-61.2\,(12.6)$	$-1.37\,(31)$
2	$-82.6\,(9.9)$	$-67.6\,(9.9)$	$-1.28\,(31)$
3	$-86.8\,(8.0)$	$-71.7\,(8.0)$	$-1.20\,(33)$
4	$-89.7\,(6.7)$	$-74.4\,(6.7)$	$-1.13\,(34)$
5	$-91.6\,(5.8)$	$-76.3\,(5.8)$	$-1.06\,(35)$
SEVP			
0			-0.01657
1	$-12.9\,(1.6)$	$-11.3\,(1.6)$	$-0.016\,(6)$
2	$-11.8\,(1.4)$	$-10.2\,(1.4)$	$-0.015\,(5)$
3	$-11.0\,(1.2)$	$-9.4\,(1.2)$	$-0.011\,(2)$
4	$-10.5\,(1.2)$	$-8.9\,(1.1)$	$-0.007\,(2)$
5	$-10.0\,(1.1)$	$-8.4\,(1.1)$	$-0.004\,(1)$
VPVP			
0			-0.26407
1	$-2.76\,(2)$	-3.37	-0.263
2	-2.70	-3.30	-0.261
3	-2.65	-3.24	-0.260
4	-2.61	-3.20	-0.259
5	-2.58	-3.16	-0.258

V. NUCLEAR SIZE AND POLARIZABILITY

It is customary in the literature to consider separately the finite nuclear size (fns) effect (also known as the elastic part of the nuclear structure) and the nuclear polarizability (also known as the inelastic nuclear structure). To a large extent, the separate treatment is due to the fact that the fns correction can be obtained numerically from the Dirac equation, whereas calculations of the nuclear polarizability are much more complicated. However, it was shown [13, 55, 56] that for light atoms, there is significant cancelation between the fns effects and the polarizability corrections. Moreover, it turned out that some of the nuclear model dependence of the individual corrections cancels out in the sum. Because of this, it is desirable to keep these contributions together and address them on the same footing. We thus consider the sum of the fns correction $E_{\rm fns}$ and the polarizability correction $E_{\rm pol}$,

$$E_{\rm nucl} = E_{\rm fns} + E_{\rm pol} = \sum_{i\geq 4} E^{(i)}_{\rm nucl}\,, \tag{44}$$

where the upper index i indicates the order in $Z\alpha$.

A. $(Z\alpha)^4$ nuclear contribution

The leading-order nuclear contribution comes solely from the finite nuclear size. It is given for an arbitrary hydrogenlike system by a simple formula,

$$E^{(4)}_{\rm nucl} = E^{(4)}_{\rm fns} = \frac{2}{3}\frac{(Z\alpha)^4}{n^3}\left(\frac{m_r}{m}\right)^3 R_C^2\,\delta_{\ell 0}\,, \tag{45}$$

where R_C is the root-mean-square (rms) charge radius of the nucleus

$$R_C^2 = \int d^3r\; r^2\,\rho(r)\,, \tag{46}$$

and $\rho(r)$ is the nuclear charge distribution.

The higher-order nuclear contributions are specific for each nucleus. We start our consideration with hydrogen, which is a special case since proton is the only non-composite (one-nucleon) nucleus.

B. $(Z\alpha)^5$ nuclear contribution for hydrogen

If we assume that the nucleus has a fixed charge density distribution, then the $(Z\alpha)^5$ nuclear correction is given by the two-Coulomb exchange amplitude. The resulting fns correction is [57]

$$E^{(5)}_{\rm fns} = -\frac{1}{3}\frac{(Z\alpha)^5}{n^3}\left(\frac{m_r}{m}\right)^3 R_Z^3\,\delta_{\ell 0}\,, \tag{47}$$

where R_Z is the third Zemach moment

$$R_Z^3 = \int d^3r_1 \int d^3r_2\,\rho(r_1)\,\rho(r_2)\,|\vec{r}_1 - \vec{r}_2|^3\,. \tag{48}$$

The numerical value for the proton is $R_{pZ} \equiv R_Z({\rm H}) = 1.41(2)$ fm, which is the average of two results derived from the electron-positron scattering [58, 59].

A more detailed consideration shows, however, that a nucleus cannot generally be treated as a rigid body, because it is polarized by the surrounding electron. This gives rise to the so-called nuclear polarizability contribution. The proton polarizability correction is usually calculated as the forward two-photon exchange amplitude, expressed via dispersion relations in terms of the inelastic scattering amplitude, which in turn is accessible in experiments.

The recent evaluation of the proton $(Z\alpha)^5$ nuclear contribution [14] yields the result of $-0.1092\,(120)$ kHz for the hydrogen $1S$ state, which agrees with the previous (elastic + polarizability) value adopted by CODATA [7] of $-0.10(1)$ kHz. The result [14] can be conveniently parameterized in terms of the effective proton radius R_{pF}, which is introduced in analogy with Eq. (47),

$$E^{(5)}_{\rm nucl}({\rm H}) = -\frac{1}{3}\frac{(Z\alpha)^5}{n^3}\left(\frac{m_r}{m}\right)^3 R_{pF}^3\,\delta_{\ell 0}\,, \tag{49}$$

with

$$R_{pF} = 1.947\,(75)\ {\rm fm}\,. \tag{50}$$

We note that for the proton there is no cancelation between the elastic and polarizability contributions, in contrast to the composite nuclei.

TABLE V: Coefficients of the $Z\alpha$ expansion of the nuclear recoil correction in Eq. (36).

Term		$1S$	$2S$	$2P_{1/2}$
D_{51}	$\frac{1}{3}\,\delta_{\ell 0}$	$\frac{1}{3}$	$\frac{1}{3}$	0
D_{50}	$-\frac{8}{3}\,\ln k_0 + \frac{14}{3}\left[1 - \frac{1}{42} - \frac{1}{2n} + \ln\frac{2}{n} + \psi(n+1) - \psi(1)\right]\delta_{\ell 0}$ $-\frac{7}{3}\,[l(l+1)(2l+1)]^{-1}(1-\delta_{\ell 0})$	2.165 899 582	2.890 835 841	−0.308 844 332
D_{60}	$\left(4\ln 2 - \frac{7}{2}\right)\pi\,\delta_{\ell 0} + 2\pi\left[3 - \frac{l(l+1)}{n^2}\right]\,[(4l^2-1)(2l+3)]^{-1}(1-\delta_{\ell 0})$	−2.285 229 926	−2.285 229 926	1.047 197 551

TABLE VI: Numerical results for the recoil higher-order remainder function in Eq. (36).

Z	$1S$	$2S$	$2P_{1/2}$
1	9.720 (3)	14.899 (3)	1.509 7 (2)
2	10.390 (1)	15.010 (1)	1.307 39 (5)
3	10.4803 (9)	14.7806 (9)	1.192 04 (2)
4	10.4155 (6)	14.4926 (6)	1.112 68 (2)
5	10.2944 (4)	14.2013 (4)	1.053 21 (2)

C. $(Z\alpha)^5$ nuclear contribution for composite nuclei

For compound nuclei consisting of several nucleons, the Zemach fns correction (47) cancels out in a sum with the corresponding nuclear structure contribution [13]. However, it survives in the contribution induced by the interaction with individual nucleons. In the result, we write the total nuclear structure correction $E^{(5)}_{\rm nucl}$ (known also as the two-photon exchange correction) for a composite nuclei as

$$E^{(5)}_{\rm nucl} = E^{(5)}_{\rm pol} - \frac{1}{3}\,\frac{\alpha^2(Z\alpha)^3}{n^3}\left[Z\,R^3_{pF} + (A-Z)\,R^3_{nF}\right]\delta_{\ell 0}\,, \tag{51}$$

where the first term $E^{(5)}_{\rm pol}$ is the intrinsic nuclear polarizability and the second term is the contribution of individual nucleons. In the above equation, R_{pF} is the effective proton radius given in Eq. (50), R_{nF} is an analogous effective radius for the neutron, and A is the mass number. We extract R_{nF} from the calculation of Tomalak (Table II of Ref. [14]), with the result

$$R_{nF} = 1.43\,(16)\ {\rm fm}\,. \tag{52}$$

The nuclear polarizability correction $E^{(5)}_{\rm pol}$ is dominated by the electric dipole excitations and is given by [13, 56, 60]

$$\begin{aligned} E^{(5)}_{\rm pol} = &-\alpha^2\,\phi^2(0)\,\frac{2}{3}\left\langle\phi_N\middle|\,\vec{d}\,\frac{1}{H_N - E_N}\left[\frac{19}{6}\right.\right. \\ &\left.\left. + 5\,\ln\frac{2\,(H_N-E_N)}{m}\right]\vec{d}\,\middle|\phi_N\right\rangle \\ &-\frac{\pi}{3}\,\alpha^2\,\phi^2(0)\,\sum_{i,j=1}^{Z}\langle\phi_N||\vec{R}_i - \vec{R}_j|^3|\phi_N\rangle \\ &+ \text{many small corrections}\,, \end{aligned} \tag{53}$$

where $\vec{d}$ is the electric dipole operator divided by the elementary charge, H_N and E_N are the nuclear Hamiltonian and its eigenvalue, ϕ_N and ϕ are the nuclear and electronic wave functions, and $\vec{R}_i$ is the position vector of ith proton in the nucleus. The second term in Eq. (53) is the remainder of the Zemach fns correction (47) for a composite nuclei.

For atoms with $Z \le 5$, the nuclear polarizability correction has been investigated only for deuterium, helium, and some neutron-rich isotopes of Li and Be. For deuteron, the two-photon nuclear polarizability was calculated in Ref. [55] and recently reanalysed in Ref. [13],

$$E^{(5)}_{\rm pol}({\rm D}) = -21.78\,\frac{\delta_{\ell 0}}{n^3}\,h\,{\rm kHz} \pm 1\%\,. \tag{54}$$

For helium, the nuclear polarizability correction was calculated in Ref. [61], with the result

$$E^{(5)}_{\rm pol}({}^4{\rm He}) = -32.1\,\frac{\delta_{\ell 0}}{n^3}\,h\,{\rm kHz} \pm 10\%\,, \tag{55}$$

$$E^{(5)}_{\rm pol}({}^3{\rm He}) = -55.2\,\frac{\delta_{\ell 0}}{n^3}\,h\,{\rm kHz} \pm 10\%\,. \tag{56}$$

For stable isotopes with $Z = 3$, 4, and 5, we use the following estimate

$$E^{(5)}_{\rm pol} \approx -\frac{E_{\rm fns}}{1000} \pm 100\%\,, \tag{57}$$

which was obtained in Ref. [17] basing on an analysis of available results throughout the whole Z sequence.

D. $(Z\alpha)^6$ nuclear contribution

The $(Z\alpha)^6$ nuclear contribution arises from the three-photon exchange between electron and the nucleus. The corresponding fns correction is known in the nonrecoil limit and is given for the nS and $nP_{1/2}$ ($\kappa = 1$) states by [13, 57]

$$\begin{aligned} E^{(6)}_{\rm fns} = \frac{(Z\alpha)^6}{n^3}\,R_C^2\Bigg\{ &-\frac{2}{3}\left[\frac{9}{4n^2} - 3 - \frac{1}{n}\right. \\ &\left. + 2\,\gamma_E - \ln\frac{n}{2} + \Psi(n) + \ln\left(mR_{C2}\,Z\,\alpha\right)\right]\delta_{\ell 0} \\ &+ \frac{1}{6}\left(1 - \frac{1}{n^2}\right)\delta_{\kappa 1}\Bigg\}\,, \end{aligned} \tag{58}$$

where R_{C2} is the effective nuclear charge radii that encodes the high-momentum contribution (for exact definition see Ref. [13]). The effective nuclear radii R_{C2} has the numerical value close to R_C and depends on the model of the nuclear charge distribution. We use the result obtained in Ref. [13] for the exponential model,

$$R_{C2}/R_C = 1.068\,497\,, \tag{59}$$

which does not depend on nuclear charge. It was shown in Ref. [13] that the dependence on R_{C2} in Eq. (58) cancels out in the sum with the corresponding nuclear polarizability correction, so the model dependence of R_{C2} does not contribute to the uncertainty.

The $(Z\alpha)^6$ nuclear polarizability is practically unknown for the electronic atoms. The only available results are estimates from Ref. [13] for hydrogen

$$E^{(6)}_{\rm pol}({\rm H}) = 0.393\,\frac{\delta_{\ell 0}}{n^3}\;h\,{\rm kHz} \pm 100\%\,, \tag{60}$$

and deuterium

$$E^{(6)}_{\rm pol}({\rm D}) = -0.541\,\frac{\delta_{\ell 0}}{n^3}\;h\,{\rm kHz} \pm 75\%\,. \tag{61}$$

It is remarkable that for hydrogen, the three-photon nuclear polarizability dominates over the two-photon polarizability. The reason for this is that $E^{(6)}_{\rm pol} \propto (Z\alpha)^6\,R_C^2$ whereas $E^{(5)}_{\rm nucl}({\rm H}) \propto (Z\alpha)^5\,R_C^3$, so that the two-photon exchange is effectively suppressed by a parameter $mR_C/(Z\alpha) \ll 1$. For all atoms other than hydrogen, the two-photon exchange is dominated by the electric dipole polarizability $\propto (Z\alpha)^5\,R_C^2$ and, therefore, the three-photon polarizability is smaller than the two-photon one, as usually expected. We estimate the uncertainty due to the unknown three-photon nuclear polarizability for nuclei with $Z = 2-5$ to be 10% of the corresponding two-photon polarizability.

E. Radiative fns correction

The leading radiative fns correction is of order $\alpha(Z\alpha)^5$ and nonzero only for S states (see review [22] for details),

$$E^{(5)}_{\rm fns,rad} = \frac{2}{3}\frac{\alpha(Z\alpha)^5}{n^3}\left(\frac{m_r}{m}\right)^3 R_C^2\,\left(4\ln 2 - 5\right)\delta_{\ell 0}\,. \tag{62}$$

The next-order radiative fns correction for the S states is known only partially [35, 62, 63],

$$E^{(6)}_{\rm fns,rad}(nS) = \frac{2}{3}\frac{\alpha\,(Z\alpha)^6}{\pi\,n^3}\,R_C^2\,\Big[-\frac{2}{3}\,\ln^2(Z\alpha)^{-2} + \ln^2(mR_C)\Big]\,. \tag{63}$$

In the above formula we keep only the squared logarithms and do not include some higher-order terms derived in Ref. [62], because the term $\propto \ln(Z\alpha)^{-2}$ is not known and expected to be of similar magnitude as the omitted terms. The result for the P states [35, 62, 63] is

$$E^{(6)}_{\rm fns,rad}\,(nP_{1/2}) = \frac{1}{6}\frac{\alpha\,(Z\alpha)^6}{\pi\,n^3}\,R_C^2\,\left(1-\frac{1}{n^2}\right) \times\left[\frac{8}{9}\,\ln(Z\alpha)^{-2} - \frac{8}{9}\ln 2 + \frac{166}{135} + \frac{4n^2}{n^2-1}\mathcal{N}(nP)\right]. \tag{64}$$

The uncertainty of Eqs. (63) and (64) was evaluated by comparing with results of the more complete treatment [63].

F. Nuclear self-energy

The nuclear self-energy correction was derived in Ref. [64], with the result

$$E_{\rm NSE} = \left(\frac{m}{M}\right)^2\,\frac{4Z(Z\alpha)^5}{3\pi n^3} \times\left[\ln\left(\frac{M}{m(Z\alpha)^2}\right)\,\delta_{\ell 0} - \ln k_0(n,l)\right]. \tag{65}$$

It should be noted that there is some ambiguity associated with this correction since the nuclear self-energy contributes not only to the Lamb shift but to the nuclear charge radius and the nuclear magnetic moment. Specifically, addition of an arbitrary constant in the brackets of Eq. (65) is equivalent to changing the definition of the nuclear charge radius. This implies that the presently used definition of the nuclear charge radius (through the slope of the Sachs form-factor) is ambiguous on the level of a constant in the brackets of Eq. (65). This issue was pointed out in Ref. [64] (together with the suggestion for a rigorous definition of the nuclear charge radius) but did not attracted attention of the community up to now. In order to quantify this ambiguity, we ascribe to $E_{\rm NSE}$ an uncertainty of 0.5 in the square brackets, as in Ref. [54]. The numerical value of this uncertainty is 0.2 kHz for the hydrogen $1S$ state, which can be disregarded at present but might become relevant in the future.

VI. NUMERICAL RESULTS

In order to obtain numerical results for the Lamb shift and the transition energies, we need to specify values of fundamental constants and nuclear parameters. In the present review we use the charge radii of the proton and the deuteron as derived from the muonic atoms [2, 65] ($R_p = 0.84087\,(39)$ fm and $R_d = 2.12562\,(78)$ fm) and the corresponding value of the Rydberg constant from Ref. [16],

$$c\,R_\infty = 3\,289\,841\,960\,248.9\,(3.0)\;{\rm kHz}\,. \tag{66}$$

It should be mentioned that the exact values of R_p, R_d, and R_∞ are under debates at present. In particular, the Rydberg constant of Eq. (66) differs from the value recommended by

CODATA 2014 [7] by $5.5\,\sigma$. On the level of the present experimental accuracy, this controversy is relevant only for hydrogen and deuterium and can be disregarded for heavier atoms.

The nuclear charge radii for elements with $Z > 1$ are taken as follows. For ^{3}He and ^{4}He, we use values by Sick [66, 67]; for ^{6}Li and ^{7}Li isotopes, values from Ref. [68]; for other atoms, values from Ref. [69]. The nuclear masses are taken for hydrogen from Ref. [70], for deuterium and helium isotopes from Ref. [7], and for all other nuclei from Ref. [71]. Nuclear magnetic moments are taken from Ref. [72]. The fine-structure constant is [7]

$$\alpha = 1/137.035\,999\,139\,(31)\,. \tag{67}$$

The individual contributions to the Lamb shift for two experimentally most interesting cases, H and He^+, are listed in Table VII. The results for the QED and the leading fns correction are presented in the nonrecoil limit (i.e., with $m_r \to 1$). The contribution due to the reduced mass in all formulas is summed up and tabulated separately as the relativistic reduced mass (RRM) correction. The uncertainty of the fns correction is due to the uncertainty of the nuclear charge radius R_C, whose values are specified in the table. The total results for the Lamb shift E_L are given with two uncertainties. The first one is the theoretical uncertainty, whereas the second one comes from the uncertainty of the nuclear charge radius.

We observe that for the hydrogen Lamb shift, the theoretical uncertainty is twice larger than the uncertainty due to the proton charge radius (as extracted from muonic hydrogen). The two largest theoretical uncertainties come from (i) the two-loop self-energy and (ii) the three-loop QED correction. As compared to the previous CODATA review [7], the main change is due to our reanalysis of the two-loop QED effects; it shifted the theoretical value by one half of the previous uncertainty and improved the accuracy by a factor of 1.5.

For helium, the uncertainty of the Lamb shift is presently dominated by the uncertainty from the nuclear radius. But this is likely to change once the results of the muonic helium experiment are evaluated [5, 8].

Table VIII presents theoretical results for the $2S$–$1S$ and $2S$–$2P_{1/2}$ transition energies in hydrogen and light hydrogen-like ions. Theoretical predictions are given with two uncertainties. The first one is the theoretical uncertainty, whereas the second one is induced by uncertainties of nuclear radii and masses. The uncertainty due to the Rydberg constant R_∞ is not included. Theoretical predictions are compared with available experimental results for the $2S$–$2P_{1/2}$ Lamb shift in hydrogen, helium and lithium. We do not present a comparison with the hydrogen $1S$–$2S$ experimental results [73, 74] since the value of the Rydberg constant (66) is derived from the comparison of theory and these experiments. For the same reason we do not include the uncertainty due to Rydberg constant in the theoretical predictions.

In summary, theoretical calculations of the Lamb shift in hydrogen and light hydrogen-like ions are required for the determination of the Rydberg constant. In the present work we summarized the present status and recent developments of theoretical calculations of QED and nuclear effects, critically evaluating uncertainties of all contributions.

Acknowledgments

The authors are grateful to A. Kramida for pointing out a mistake in an early version of the manuscript. V.A.Y. acknowledges support by the Ministry of Education and Science of the Russian Federation Grant No. 3.5397.2017/6.7. K.P. acknowledges support by the National Science Center (Poland) Grant No. 2017/27/B/ST2/02459. V.P. acknowledges support from the Czech Science Foundation - GAČR (Grant No. P209/18-00918S).

[1] R. Pohl, A. Antognini, F. Nez, F. D. Amaro, F. Biraben, J. a. M. R. Cardoso, D. S. Covita, A. Dax, S. Dhawan, L. M. P. Fernandes, A. Giesen, T. Graf, T. W. Hänsch, P. Indelicato, L. Julien, C.-Y. Kao, P. Knowles, E.-O. Le Bigot, Y.-W. Liu, J. A. M. Lopes, L. Ludhova, C. M. B. Monteiro, F. Mulhauser, T. Nebel, P. Rabinowitz, J. M. F. dos Santos, L. A. Schaller, K. Schuhmann, C. Schwob, D. Taqqu, J. a. F. C. A. Veloso, F. Kottmann, *Nature (London)* **2010**, *466*, 213.

[2] A. Antognini, F. Nez, K. Schuhmann, F. D. Amaro, F. Biraben, J. M. R. Cardoso, D. S. Covita, A. Dax, S. Dhawan, M. Diepold, L. M. P. Fernandes, A. Giesen, A. L. Gouvea, T. Graf, T. W. Hänsch, P. Indelicato, L. Julien, C.-Y. Kao, P. Knowles, F. Kottmann, E.-O. L. Bigot, Y.-W. Liu, J. A. M. Lopes, L. Ludhova, C. M. B. Monteiro, F. Mulhauser, T. Nebel, P. Rabinowitz, J. M. F. dos Santos, L. A. Schaller, C. Schwob, D. Taqqu, J. F. C. A. Veloso, J. Vogelsang, R. Pohl, *Science* **2013**, *339*, 417.

[3] A. Beyer, L. Maisenbacher, A. Matveev, R. Pohl, K. Khabarova, A. Grinin, T. Lamour, D. C. Yost, T. W. Hänsch, N. Kolachevsky, T. Udem, *Science* **2017**, *358*, 79.

[4] H. Fleurbaey, S. Galtier, S. Thomas, M. Bonnaud, L. Julien, F. Biraben, F. Nez, M. Abgrall, J. Guéna, *Phys. Rev. Lett.* **2018**, *120*, 183001.

[5] R. Pohl, talk at *670th WE-Heraeus-Seminar Fundamental Constants: Basic Physics and Units*, May 13-18, Bad Honnef, Germany, 2018.

[6] E. Hessels, talk at *Precision Measurements and Fundamental Physics: The Proton Radius Puzzle and Beyond*, July 23-27, 2018 Mainz, Germany, 2018.

[7] P. J. Mohr, D. B. Newell, B. N. Taylor, *Rev. Mod. Phys.* **2016**, *88*, 035009.

[8] S. Schmidt, M. Willig, J. Haack, R. Horn, A. Adamczak, M. A. Ahmed, F. D. Amaro, P. Amaro, F. Biraben, P. Carvalho, T.-L. Chen, L. M. P. Fernandes, T. Graf, M. Guerra, T. W. Hänsch, M. Hildebrandt, Y.-C. Huang, P. Indelicato, L. Julien, K. Kirch, A. Knecht, F. Kottmann, J. J. Krauth, Y.-W. Liu, J. Machado, M. Marszalek, C. M. B. Monteiro, F. Nez, J. Nuber, D. N. Patel, E. Rapisarda, J. M. F. dos Santos, J. P. Santos, P. A. O. C. Silva, L. Sinkunaite, J.-T. Shy, K. Schuhmann, I. Schulthess, D. Taqqu, J. F. C. A. Veloso, L.-B. Wang, M. Zeyen, A. Antognini, R. Pohl, *arXiv:1808.07240* **2018**.

[9] T. Udem, private communication, 2018.

[10] K. Eikema, private communication, 2017.

TABLE VII: Individual contributions to the Lamb shift E_L, in MHz. Abbreviations are as follows: "SE" is the one-loop self-energy, "Ue" is the Uehling one-loop vacuum polarization, "WK" is the Wichmann-Kroll one-loop vacuum-polarization, "Ue(μhad)" is the Uehling muon and hadronic vacuum polarization, "SESE" is the two-loop self-energy, "SEVP" is the electron self-energy with vacuum-polarization insertions, "VPVP" is the two-loop vacuum-polarization, "QED(ho)" is the three-loop QED correction, "RRM" is the relativistic reduced mass correction (see text), "REC" is the recoil correction $E_{\rm REC}$, "REC(ho)" is the sum of higher-order recoil corrections $E_{\rm REC,2}$, $E_{\rm REC,hfs}$, and $E_{\rm RREC}$, "FNS" is the leading-order fns correction $E^{(4)}_{\rm nucl}$, "NUCL5" is the $(Z\alpha)^5$ nuclear correction $E^{(5)}_{\rm nucl}$, "NUCL6" is the $(Z\alpha)^6$ nuclear correction $E^{(6)}_{\rm nucl}$, "FNS(rad)" is the radiative fns correction $E_{\rm fns,rad}$, "NSE" is the nuclear self-energy correction $E_{\rm NSE}$.

	$1S$	$2S$	$2P_{1/2}$
$Z=1,\ ^1$H , $R_C = 0.840\,87\,(39)$ fm, $M/m = 1\,836.152\,673\,346\,(81)$			
SE	8 396.453 556 (1)	1 072.958 455	−12.858 661 (1)
Ue	−215.170 186	−26.897 303	−0.000 347
WK	0.002 415	0.000 302	0
Ue(μhad)	−0.008 48 (8)	−0.001 06 (1)	0
SESE	2.335 0 (13)	0.292 48 (16)	0.027 253 (4)
SEVP	0.288 39 (16)	0.036 015 (20)	−0.001 241
VPVP	−1.895 224	−0.236 911	−0.000 003
QED(ho)	0.001 83 (96)	0.000 23 (12)	−0.000 216
RRM	−12.765 917	−1.633 931	0.011 741
REC	2.402 830	0.340 469	−0.016 656
REC(ho)	0.013 16 (74)	−0.003 227 (92)	−0.001 335 (4)
FNS	1.107 6 (10)	0.138 45 (13)	0
NUCL5	−0.000 109 (1)	−0.000 014	0
NUCL6	0.001 07 (39)	0.000 140 (49)	0.000 001
FNS(rad)	−0.000 135 (1)	−0.000 017	0
NSE	0.004 63 (16)	0.000 585 (20)	0.000 001 (20)
Total	8 172.770 4 (18)(10)	1 044.994 66 (23)(13)	−12.839 463 (21)(0)
$Z=2,\ ^4$He$^+$, $R_C = 1.6810\,(40)$ fm, $M/m = 7\,294.299\,541\,36\,(24)$			
SE	111 054.170 69 (1)	14 257.035 60 (2)	−204.794 17 (2)
Ue	−3 415.099 45	−426.952 77	−0.022 109
WK	0.152 06	0.019 01	0.000 002
Ue(μhad)	−0.135 7 (12)	−0.016 97 (15)	0
SESE	32.569 (64)	4.095 9 (80)	0.440 50 (25)
SEVP	4.956 8 (88)	0.617 9 (11)	−0.020 086
VPVP	−30.047 28	−3.756 393 (1)	−0.000 210
QED(ho)	0.029 (31)	0.003 6 (38)	−0.003 468
RRM	−41.915 19	−5.393 65	0.046 950
REC	17.676 28	2.533 38	−0.130 835
REC(ho)	−0.121 (10)	−0.020 0 (13)	0.000 549
FNS	70.82 (34)	8.853 (42)	0
NUCL5	−0.034 6 (32)	−0.004 33 (40)	0
NUCL6	0.152 0 (35)	0.020 66 (43)	0.000 354
FNS(rad)	−0.017 43 (39)	−0.002 179 (50)	0.000 007
NSE	0.018 75 (65)	0.002 372 (82)	0.000 005 (82)
Total	107 693.18 (7)(34)	13 837.035 (9)(42)	−204.482 51 (26)(0)

[11] V. A. Yerokhin, V. M. Shabaev, *Phys. Rev. Lett.* **2015**, *115*, 233002.
[12] V. A. Yerokhin, V. M. Shabaev, *Phys. Rev. A* **2016**, *93*, 062514.
[13] K. Pachucki, V. Patkóš, V. A. Yerokhin, *Phys. Rev. A* **2018**, *97*, 062511.
[14] O. Tomalak, *arXiv:1808.09204*, **2018**.
[15] U. D. Jentschura, V. Yerokhin, *Phys. Rev. A* **2006**, *73*, 062503.
[16] M. Horbatsch, E. A. Hessels, *Phys. Rev. A* **2016**, *93*, 022513.
[17] V. A. Yerokhin, V. M. Shabaev, *J. Phys. Chem. Ref. Data* **2015**, *44*, 033103.
[18] J. R. Sapirstein, D. R. Yennie, in *Quantum Electrodynamics*, ed. by T. Kinoshita, p. 560 – 672, World Scientific, Singapore, 1990.
[19] W. A. Barker, F. N. Glover, *Phys. Rev.* **1955**, *99*, 317.
[20] S. J. Brodsky, R. G. Parsons, *Phys. Rev.* **1967**, *163*, 134.
[21] S. R. Lundeen, F. M. Pipkin, *Phys. Rev. Lett.* **1981**, *46*, 232.
[22] M. I. Eides, H. Grotch, V. A. Shelyuto, *Phys. Rep.* **2001**, *342*, 63.
[23] U. D. Jentschura, P. J. Mohr, G. Soff, *Phys. Rev. Lett.* **1999**, *82*, 53.
[24] U. D. Jentschura, P. J. Mohr, G. Soff, *Phys. Rev. A* **2001**, *63*, 042512.
[25] U. D. Jentschura, A. Czarnecki, K. Pachucki, *Phys. Rev. A* **2005**, *72*, 062102.
[26] N. L. Manakov, A. A. Nekipelov, *Vestnik Voronezhskogo gosudarstvennogo universiteta* **2012**, *2*, 53, [in Russian,

TABLE VIII: Theoretical transition energies of light hydrogen-like atoms (in GHz), in comparison with available experimental results.

Z		R_C [fm]	M/m	$2S$–$1S$	$2S$–$2P_{1/2}$
1	^{1}H	0.84087 (39)	1 836.152 673 346 (81)	2 466 061.413 1869 (18)(10)	1.057 834 12 (23)(13) 1.057 847 (9)[a]
1	^{2}D	2.12562 (78)	3 670.482 967 85 (13)	2 466 732.407 5345 (17)(52)	1.059 219 91 (21)(65)
2	^{4}He$^+$	1.6810 (40)	7 294.299 541 36 (24)	9 868 561.006 31 (7)(34)	14.041 517 (9)(42) 14.041 13 (17)[b]
2	^{3}He$^+$	1.973 (16)	5 495.885 279 22 (27)	9 868 118.3826 (1)(16)	14.043 96 (1)(20)
3	^{6}Li^{2+}	2.589 (39)	10 961.898 642 0 (83)	22 206 430.550 (1)(26)	62.734 2 (1)(32) 62.765 (21)[c]
3	^{7}Li^{2+}	2.444 (42)	12 786.392 271 (11)	22 206 719.625 (1)(26)	62.723 1 (1)(33)
4	^{9}Be^{3+}	2.519 (12)	16 424.205 51 (16)	39 482 224.239 (4)(24)	179.771 9 (5)(30)
5	^{11}B^{4+}	2.406 (29)	20 063.737 33 (78)	61 697 635.70 (1)(14)	404.523 (1)(17)

[a] Experiment by Lundeen and Pipkin [75]; the original result is shifted by 1.88 kHz due to the off-diagonal hfs correction, in order to comply with the present definition of the Lamb shift,
[b] Experiment by van Wijngaarden et al. [76],
[c] Experiment by Leventhal [77].

http://www.vestnik.vsu.ru/pdf/physmath/2012/02/2012-02-07.pdf].

[27] M. I. Eides, V. A. Shelyuto, *Phys. Rev. A* **1995**, *52*, 954.
[28] S. G. Karshenboim, *J. Phys. B* **1995**, *28*, L77.
[29] J. L. Friar, J. Martorell, D. W. L. Sprung, *Phys. Rev. A* **1999**, *59*, 4061.
[30] K. Pachucki, *Phys. Rev. A* **2001**, *63*, 042503.
[31] K. Pachucki, U. D. Jentschura, *Phys. Rev. Lett.* **2003**, *91*, 113005.
[32] A. Czarnecki, U. D. Jentschura, K. Pachucki, *Phys. Rev. Lett.* **2005**, *95*, 180404.
[33] M. Dowling, J. Mondéjar, J. H. Piclum, A. Czarnecki, *Phys. Rev. A* **2010**, *81*, 022509.
[34] A. Czarnecki, R. Szafron, *Phys. Rev. A* **2016**, *94*, 060501.
[35] U. D. Jentschura, *J. Phys. A* **2003**, *36*, L229.
[36] U. D. Jentschura, *Phys. Rev. A* **2006**, *74*, 062517.
[37] S. G. Karshenboim, V. G. Ivanov, *Phys. Rev. A* **2018**, *98*, 022522.
[38] V. A. Yerokhin, P. Indelicato, V. M. Shabaev, *Phys. Rev. Lett.* **2003**, *91*, 073001.
[39] V. A. Yerokhin, P. Indelicato, V. M. Shabaev, *Eur. Phys. J. D* **2003**, *25*, 203.
[40] V. A. Yerokhin, *Phys. Rev. A* **2009**, *80*, 040501(R).
[41] V. A. Yerokhin, *Phys. Rev. A* **2018**, *97*, 052509.
[42] V. A. Yerokhin, P. Indelicato, V. M. Shabaev, *Phys. Rev. A* **2008**, *77*, 062510.
[43] S. Laporta, E. Remiddi, *Phys. Lett.* **1996**, *B 379*, 283.
[44] K. Melnikov, T. van Ritbergen, *Phys. Rev. Lett.* **2000**, *84*, 1673.
[45] M. I. Eides, V. A. Shelyuto, *Phys. Rev. A* **2004**, *70*, 022506.
[46] M. I. Eides, V. A. Shelyuto, *Can. J. Phys.* **2007**, *85*, 509.
[47] K. Pachucki, S. G. Karshenboim, *Phys. Rev. A* **1999**, *60*, 2792.
[48] K. Melnikov, A. S. Yelkhovsky, *Phys. Lett. B* **1999**, *458*, 143.
[49] G. W. Erickson, *J. Phys. Chem. Ref. Dat.* **1977**, *6*, 831.
[50] K. Pachucki, S. Karshenboim, *J. Phys. B* **1995**, *28*, L221.
[51] K. Pachucki, H. Grotch, *Phys. Rev. A* **1995**, *51*, 1854.
[52] A. Czarnecki, K. Melnikov, *Phys. Rev. Lett.* **2001**, *87*, 013001.
[53] M. I. Eides, H. Grotch, V. A. Shelyuto, *Phys. Rev. A* **2001**, *63*, 052509.
[54] P. J. Mohr, B. N. Taylor, D. B. Newell, *Rev. Mod. Phys.* **2012**, *84*, 1527.
[55] J. L. Friar, G. L. Payne, *Phys. Rev. C* **1997**, *56*, 619.
[56] K. Pachucki, *Phys. Rev. Lett.* **2011**, *106*, 193007.
[57] J. L. Friar, *Ann. Phys. (NY)* **1979**, *122*, 151.
[58] J. Friar, I. Sick, *Phys. Lett. B* **2004**, *579*, 285.
[59] M. O. Distler, J. C. Bernauer, T. Walcher, *Phys. Lett. B* **2011**, *696*, 343.
[60] O. Hernandez, C. Ji, S. Bacca, N. N. Dinur, N. Barnea, *Phys. Lett. B* **2014**, *736*, 344.
[61] K. Pachucki, A. M. Moro, *Phys. Rev. A* **2007**, *75*, 032521.
[62] A. I. Milstein, O. P. Sushkov, I. S. Terekhov, *Phys. Rev. A* **2003**, *67*, 062111.
[63] V. A. Yerokhin, *Phys. Rev. A* **2011**, *83*, 012507.
[64] K. Pachucki, *Phys. Rev. A* **1995**, *52*, 1079.
[65] R. Pohl, F. Nez, L. M. P. Fernandes, F. D. Amaro, F. Biraben, J. M. R. Cardoso, D. S. Covita, A. Dax, S. Dhawan, M. Diepold, A. Giesen, A. L. Gouvea, T. Graf, T. W. Hänsch, P. Indelicato, L. Julien, P. Knowles, F. Kottmann, E.-O. Le Bigot, Y.-W. Liu, J. A. M. Lopes, L. Ludhova, C. M. B. Monteiro, F. Mulhauser, T. Nebel, P. Rabinowitz, J. M. F. dos Santos, L. A. Schaller, K. Schuhmann, C. Schwob, D. Taqqu, J. F. C. A. Veloso, A. Antognini, *Science* **2016**, *353*, 669.
[66] I. Sick, *Phys. Rev. C* **2008**, *77*, 041302.
[67] I. Sick, *arXiv:1505.06924* **2015**.
[68] W. Nörtershäuser, T. Neff, R. Sánchez, I. Sick, *Phys. Rev. C* **2011**, *84*, 024307.
[69] I. Angeli, K. Marinova, *At. Dat. Nucl. Dat. Tabl.* **2013**, *99*, 69.
[70] F. Heiße, F. Köhler-Langes, S. Rau, J. Hou, S. Junck, A. Kracke, A. Mooser, W. Quint, S. Ulmer, G. Werth, K. Blaum, S. Sturm, *Phys. Rev. Lett.* **2017**, *119*, 033001.
[71] M. Wang, G. Audi, A. H. Wapstra, F. G. Kondev, M. MacCormick, X. Xu, B. Pfeiffer, *Chin. Phys. C* **2012**, *36*, 1603.
[72] N. J. Stone, *At. Dat. Nucl. Dat. Tabl.* **2005**, *90*, 75.
[73] C. G. Parthey, A. Matveev, J. Alnis, R. Pohl, T. Udem, U. D. Jentschura, N. Kolachevsky, T. W. Hänsch, *Phys. Rev. Lett.* **2010**, *104*, 233001.
[74] A. Matveev, C. G. Parthey, K. Predehl, J. Alnis, A. Beyer, R. Holzwarth, T. Udem, T. Wilken, N. Kolachevsky, M. Abgrall, D. Rovera, C. Salomon, P. Laurent, G. Grosche, O. Terra, T. Legero, H. Schnatz, S. Weyers, B. Altschul, T. W. Hänsch, *Phys. Rev. Lett.* **2013**, *110*, 230801.
[75] S. R. Lundeen, F. M. Pipkin, *Metrologia* **1986**, *22*, 9.
[76] A. van Wijngaarden, F. Holuj, G. W. F. Drake, *Phys. Rev. A* **2000**, *63*, 012505.
[77] M. Leventhal, *Phys. Rev. A* **1975**, *11*, 427.